Fourier transform

The Fourier integral has numerous applications in problems of mechanics and physics. At the same time, a closely related, complex-valued mathematical object called *Fourier transform.*

Let's figure out what it is first.

Fourier integral of a function $f(t)$, absolutely integrable on any interval of the real axis, piecewise continuous $\forall x \in (-\infty, +\infty)$ and having for any real x one-sided derivatives, matches the function

$$Y(x) = \frac{1}{\pi} \int_{0}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(t) \cos \omega (x-t) dt .$$

Taking into account that the expression $\int_{-\infty}^{+\infty} f(t) \cos \omega (x-t) dt$ there is an even function on a variable ω, and the expression $\int_{-\infty}^{+\infty} f(t) \sin \omega (x-t) dt$ odd, we can write that

$$Y(x) = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-A}^{+A} d\omega \int_{-\infty}^{+\infty} f(t) \cos \omega (x-t) dt$$ And $$0 = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-A}^{+A} d\omega \int_{-\infty}^{+\infty} f(t) \sin \omega (x-t) dt .$$

If we multiply both sides of the second equality by an imaginary unit and then add both equalities term by term, then *in the limit*, using Euler's formula we get.

$$Y(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}f(t)\cos\omega(x-t)dt+\frac{i}{2\pi}\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}f(t)\sin\omega(x-t)dt=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}f(t)\big(\cos\omega(x-t)+i\sin\omega(x-t)\big)dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}f(t)e^{i\omega(x-t)}dt\,.$$

And, if we also assume that the function $f(x)$ is continuous, then $Y(x)=f(x)$, then the last equality can be written in symmetric form

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{i\omega x}d\omega\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt\,. \qquad (1)$$

Here we will have to make a “lyrical digression” and remember that the outer integral in formula (1) (that is, the integral over the variable ω) is not just an improper integral with two singular points $+\infty$ And $-\infty$. Namely, the passage to the limit at both singular points was done “synchronously,” which is prohibited by definition in an ordinary improper integral.

In other words, here we are dealing with some special type of improper integrals. This feature is explained

Example 1. Find $I = \int\limits_{-1}^{1} \frac{dx}{x}$.

This integral is improper, having two singular points: "$+0$" And "-0". For it to converge, it is necessary that it converges in each of them. Take, for example, "$+0$". We have

$$I_{+0} = \int\limits_{\to +0}^{1} \frac{dx}{x} = \lim_{\varepsilon \to +0} \int\limits_{\varepsilon}^{1} \frac{dx}{x} = \lim_{\varepsilon \to +0} \ln x \Big|_{\varepsilon}^{1} = -\lim_{\varepsilon \to +0} \ln \varepsilon = +\infty$$

Integral I_{+0} diverges, which means the integral also diverges I. Note that the integral also diverges I_{-0}.

Now let us perform the passage to the limit at the points "$+0$" And "-0""synchronously", using the same scheme as in deriving formula (1):

$$I_{?} = \lim_{\varepsilon \to +0} \left(\int\limits_{-1}^{-\varepsilon} \frac{dx}{x} + \int\limits_{\varepsilon}^{1} \frac{dx}{x} \right) = \lim_{\varepsilon \to +0} \left(\ln |x| \Big|_{-1}^{-\varepsilon} + \ln x \Big|_{\varepsilon}^{1} \right) = 0$$

.

Whence it follows that the integral converges.

To distinguish improper integrals of this kind, Cauchy proposed calling them integrals *in the sense of the main meaning* and denoted by the symbol "v.p." (from fr. *vmain alue*). So,

$$\text{v.p.}\int_{-1}^{1}\frac{dx}{x}=0 \; .$$

It is clear that from the convergence of an improper integral in the usual sense, it follows that it converges in the sense of the principal value, but not vice versa.

To solve practical problems, it turned out to be convenient, based on formula (5), to give

Definition 1. Function

$$\hat{f}(\omega) = \text{v.p.} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

called *Fourier transform* functions $f(x)$, a function

$$\tilde{f}(\omega) = \text{v.p.} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x) e^{i\omega x} dx$$

− *inverse transformation* Fourier.

The notation is also often used

$\hat{f}(\omega) = F[f]$ And $\tilde{f}(\omega) = F^{-1}[f]$.

Example 2. Find *the opposite* Fourier transform of a function

$$f(x)=\begin{cases} 1, \text{ при } |x| \le u, \\ 0, \text{ при } |x| > u. \end{cases}$$

Solution: For $\tilde{f}(\omega)$ we have

$$\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(u)e^{i\omega x}\,dx=\frac{1}{\sqrt{2\pi}}\int\limits_{-u}^{u} e^{i\omega x}\,dx=$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{e^{i\omega u}-e^{-i\omega u}}{2i\omega}=\sqrt{\frac{2}{\pi}}\,\frac{\sin u\omega}{\omega}.$$

Example 3. Find the Fourier transform of a function $f(x)=\dfrac{1}{1+x^2}$.

For $\hat{f}(\omega)$ we have

$$\hat{f}(\omega)=F\left[\frac{1}{1+x^2}\right]=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{1}{1+x^2}\,e^{-i\omega x}\,dx=$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{\cos(\omega x)}{1+x^2}\,dx-\frac{i}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{\sin(\omega x)}{1+x^2}\,dx=$$

в силу четности косинуса и нечетности синуса, а также используя значение интеграла Лапласа, получаем

$$=\sqrt{\frac{2}{\pi}}\int\limits_{0}^{+\infty}\frac{\cos(\omega x)}{1+x^2}\,dx=\sqrt{\frac{2}{\pi}}\,\frac{\pi}{2}\,e^{-|\omega|}=\sqrt{\frac{\pi}{2}}\,e^{-|\omega|}.$$

Properties of the Fourier Transform

Let us formulate the basic properties of the Fourier transform.

1°. Let the function $f(x)$ absolutely integrable to $\mathbf{R}$, Then $\hat{f}(\omega)$ continuous and limited to $\mathbf{R}$ whose function $\lim\limits_{\omega\to\pm\infty} \hat{f}(\omega) = 0$.

2°. Let the function $f(x)$ is absolutely integrable and has finite one-sided derivatives on $\mathbf{R}$, Then

$$F^{-1}[F[f]] = F[F^{-1}[f]] = f \,.$$

3°. Linearity of the Fourier transform: if the Fourier transforms of functions f And g exist, then for any complex λ And μ equality is true

$$F[\lambda f + \mu g] = \lambda F[f] + \mu F[g] \,.$$

4°. *Fourier transform of the derivative of a function*: if derivative functions $f^{(k)}(x)\ \ k=0,1,2,\ldots n$ are continuously differentiable and absolutely integrable, then

$$\forall k=[1,n] \qquad F[f^{(k)}]=(i\omega)^k F[f].$$

At the same time $\exists C>0$ such that $|F[f]|\le \dfrac{C}{|\omega^n|}$.
In other words, than *more* the function has absolutely integrable derivatives, so *faster* its Fourier transform tends to zero at infinity and vice versa.

5°. *Derivative of the Fourier transform*: let function $f(x)$ is continuous, and functions of the form $x^k f(x)\quad \forall k=0,1,2,\ldots n$ absolutely integrable on $\mathbf{R}$, then its Fourier transform $\hat{f}(\omega)$ There is n times differentiable by $\mathbf{R}$ function. At the same time

$$\hat{f}^{(k)}(\omega)=(-i)^k F[x^k f(x)] \quad k=[1,n].$$

We illustrate the use of the properties of the Fourier transform with the following examples.

Example 4. Let $f(x)=\dfrac{1}{1+|x^5|}$. Show that

1) $\hat f(\omega)$ has on $\mathbf{R}$ continuous derivative of third order;

2). $\hat f(\omega)=O\left(\dfrac{1}{\omega^5}\right)$ at $\omega\to\infty$.

Solution:

1) Sufficient conditions for integrability (comparison test) of a function $f(x)=\dfrac{x^k}{1+|x^5|}$ essence $k=1,2,3$. So, according to the property 5⁰ $\hat f(\omega)=F[f]$ has derivatives up to the third order inclusive.

2) Function $\dfrac{1}{1+|x^5|}$ has continuous derivatives up to the fourth order inclusive and piecewise continuous *fifth* derivative of the form $120\,\text{sign}(x)$. Then from property 4⁰ we obtain the required estimate.

Example 5. On set $x \in \mathbf{R}, t \geq 0$ find function $u(x,t)$, satisfying the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ – *thermal conductivity* such that $u(x,0) = u_0(x)$.

Solution: Let us introduce the following notation: $\frac{\partial u}{\partial t} = u_t^{/}$, $\frac{\partial u}{\partial x} = u_x^{/}$, $\frac{\partial^2 u}{\partial x^2} = u_{xx}^{//}$. In addition, we will assume that the functions $u(x,t), u_x^{/}(x,t), u_{xx}^{//}(x,t)$ absolutely integrable in x on the entire real axis for each $t \ge 0$. Finally, we also agree that

$$\exists \varphi(x): \forall t \ge 0 \ \left| u_t^{/}(x,t) \right| \le \varphi(x) \text{ и } \int\limits_{-\infty}^{+\infty} \varphi(x)\, dx < +\infty . \qquad (6)$$

main idea: to both sides of the equation $u_t^{/} = u_{xx}^{//}$ apply the Fourier transform according to x, counting t parameter.

Let

$$F[u(\omega,t)] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} u(x,t) e^{-i\omega x} dx .$$

For greater clarity, we will not use the 4º property, but will use the definition of the Fourier transform directly.

Then, having integrated twice by parts and taking into account that the integrated terms vanish by virtue of (6), we obtain the equalities

$$F[u''_{xx}] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} u''_{zz} e^{-i\omega z} dz = \frac{1}{\sqrt{2\pi}} \left(u'_z e^{-i\omega z} \Big|_{-\infty}^{+\infty} + i\omega \int\limits_{-\infty}^{+\infty} u'_z e^{-i\omega z} dz \right) =$$

$$= \frac{i\omega}{\sqrt{2\pi}} \left(u e^{-i\omega z} \Big|_{-\infty}^{+\infty} + i\omega \int\limits_{-\infty}^{+\infty} u e^{-i\omega z} dz \right) = -\omega^2 \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} u e^{-i\omega z} dz = -\omega^2 F[u]$$

On the other hand, due to the assumptions made, and since in the Fourier transformed equation, ω – parameter, $F[u_t^{/}] = F\left[\frac{\partial u}{\partial t}\right] = \frac{dF}{dt}$. And thus, the original partial differential equation is reduced to an ordinary differential equation (where the unknown is the function F) type:

$$\frac{dF}{dt} = -\omega^2 F,$$

whose general solution is a family of functions $\hat{f}(\omega,t) = D(\omega)e^{-\omega^2 t}$.

The initial condition at $t=0$ in our problem is the function $u_0(x)$, whose Fourier transform is the function $D(\omega)$. This means that $D(\omega) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} u_0(z)e^{-i\omega z}dz$.

As a result, we get what we are looking for *solution of the heat equation* found by applying to the function $\hat{f}(\omega,t)$ *reverse* Fourier transforms.

Hence,

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}D(\omega)e^{i\omega x-\omega^2 t}\,d\omega$$, Where $$D(\omega)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}u_0(z)e^{-i\omega z}\,dz$$.

For example (check it yourself using the solution to problem §17, No. 8(2)), what if $u_0(x)=e^{-\frac{x^2}{2}}$, That $\hat{f}(\omega,t)=e^{-\omega^2\left(t+\frac{1}{2}\right)}$. Where does it come from that

$$u(x,t)=F^{-1}\left[e^{-\omega^2\left(t+\frac{1}{2}\right)}\right]=\frac{1}{\sqrt{2t+1}}e^{-\frac{x^2}{4t+2}}$$.