Functional sequences

When solving applied problems, one often has to face the need approximation of some functions by other, simpler ones. It is also allowed that this approximation is carried out with some error.

An example of such an approximation is, for example, the Taylor formula. However, other approaches to implementing this idea are also possible. Let's consider one of them.

Let's give

Definition 8.1	Function set $f_k(x)$ $\forall k \in \mathbb{N}$ defined by $\forall x \in X \subseteq \mathbb{R}$, we will call *a functional sequence* and will denote as $\{f_k(x)\}$.

Pointwise convergence of a functional sequence

Definition 8.2	We call the function $F(x)$ *limit function* for the function sequence $\{f_k(x)\}$, if for each numerical sequence of the form $\{f_k(x_0)\}\ \forall x_0 \in X$ occurs $$\lim_{k\to\infty} f_k(x_0) = F(x_0)\,.$$

Note that if a limit of a number sequence exists, then it is unique. Therefore, the dependence $F(x)$ is obviously a *function*.

Then you can give

Definition 8.3	Let's say what if $$\lim_{k\to\infty} f_k(x_0) = F(x_0) \quad \forall x_0 \in X\,, \qquad (8.1)$$ then the functional sequence $\{f_k(x)\}$ *converges pointwise* to $F(x)$ on the set X .

Recall that in quantifier form, condition (8.1) is formulated as follows:
$\forall x_0 \in X$ and $\forall \varepsilon > 0$ $\exists$ number $N_{x_0,\,\varepsilon}$ such,
what $\forall n \geq N_{x_0,\,\varepsilon}$ inequality is valid

$$\Big|\, f_n(x_0) - F(x_0) \,\Big| < \varepsilon\,. \qquad (8.2)$$

Here we note that the symbol $N_{x_0,\,\varepsilon}$ means ability to choose number N for each x_0 and ε in its own way, regardless of how this choice was made for other x_0 and ε.

Let we are only interested in the values of the function $F(x)$. In this case using $f_k(x)$ as an approximation of $F(x)$ it can be quite justified.

However, things are not so simple in a situation where we care the coincidence of some properties of $F(x)$ and the approximating function. For example, these properties may be continuity or differentiability.

A typical case for pointwise convergence is demonstrated by

`Problem 8.1` *Find the limit function for a functional sequence* $f_k(x) = x^k$ *on set* $x \in [0,1]$.

`Solution.` It is easy to see that in this case

$$\lim_{k\to\infty} x^k = 0, \quad \text{if} \quad x \in [0,1)\,,$$

and

$$\lim_{k\to\infty} x^k = 1, \quad \text{if} \quad x = 1\,.$$

That is,

$$F(x) = \begin{cases} 0, & \text{at} \quad x \in [0,1), \\ 1, & \text{at} \quad x = 1. \end{cases}$$

Comparison of properties of functions $F(x)$ and x^k on the segment $[0,1]$ shows that the first of them is not continuous. The second is not only continuous, but also has a derivative of any order on this segment. Fig. 1 illustrates this fact.

On the other hand, it is easy to show that for a given functional sequence on the set $[0,\alpha]$ (where $\alpha < 1$) the limit function will be infinitely differentiable.

Conclusion: with pointwise convergence, the properties of the terms functional sequence are optional coincide with the properties of the limit function.

`Solution found.`

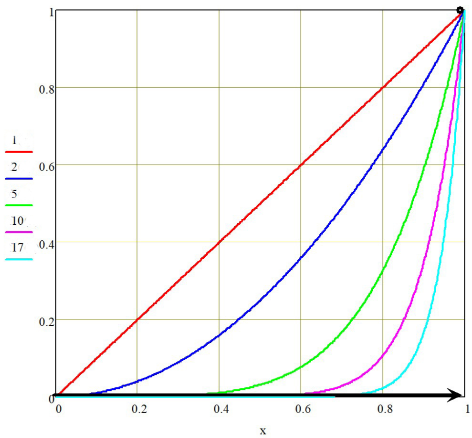

Fig. 1. Graphs of functions $f_k(x) = x^k$
for $k = 1, 2, 5, 10, 17$ to problem 8.1.

Uniform convergence of a functional sequence

Now we consider the conditions that guarantee the coincidence of properties of members functional sequence and its limit function. They are formulated by introducing a special type of convergence called *uniform convergence* on the set X.

Let us slightly change the quantifier formulation of Definition 8.3. Let's give

Definition 8.4	We will say that the functional sequence $\{f_k(x)\}$ *converges uniformly on the set* X , if $\forall\varepsilon > 0\ \ \exists N_\varepsilon \in \mathbb{N}$ such that $\forall x \in X$ and $\forall n \geq N_\varepsilon$ inequality $$\Big| f_n(x) - F(x) \Big| < \varepsilon\,.$$ is valid.

The main difference between Definition 8.4 and notation (8.2) is that in the case of uniform convergence of the functional series, number N_ε is found (selected) according to the same rule *for all* points of the set X.

In addition, uniform convergence implies pointwise convergence, but not vice versa!

Let us also agree on the following notation.

Pointwise convergence of a functional sequence $\{f_k(x)\}$ on set X to the limit function $F(x)$ we will denote it like this $f_k(x) \underset{X}{\to} F(x)$. And for uniform convergence $\{f_k(x)\}$ on set X we will use the notation $f_k(x) \underset{X}{\rightrightarrows} F(x)$.

Accordingly, cases of absence of pointwise or uniform convergence denoted by symbols $f_k(x) \underset{X}{\not\to} F(x)$. And $f_k(x) \underset{X}{\not\rightrightarrows} F(x)$.

An example of the use of these notations would be

Definition 8.5	We will say that the functional sequence $\{f_k(x)\}$ *converges unevenly on the set* X, if $f_k(x) \underset{X}{\to} F(x)$, But $f_k(x) \underset{X}{\not\rightrightarrows} F(x)$.

A natural question arises: what is the benefit of uniform convergence? The answer to this question is given by the following theoretical facts.

Theorem 8.1 **If functions $f_k(x)$ are continuous on $[a, b]$ and $\{f_k(x)\}$ converges uniformly on $[a, b]$ to the limit function $F(x)$, then $F(x)$ is continuous on $[a, b]$.**

Theorem 8.2 **If functions $f_k(x)$ are continuous on $[a, b]$ and $\{f_k(x)\}$ converges uniformly on $[a, b]$ to the limit function $F(x)$, then the functional sequence $\left\{ \int\limits_{x_0}^{x} f_k(u)\,du \right\}$ converges uniformly to the function $\int\limits_{x_0}^{x} F(u)\,du$, where $x_0 \in [a, b]$.**

Theorem 8.3 **If functions $f_k(x)$ are differentiable on $[a, b]$ and**

— **$\{f_k(x)\}$ converges at least at one point $x_0 \in [a, b]$,**

— **sequence $\{f'_k(x)\}$ converges uniformly on $[a, b]$,**

then the sequence $\{f_k(x)\}$ converges uniformly on $[a, b]$ to continuously differentiable on $[a, b]$ function $F(x)$ such that

$$F'(x) = \lim_{k\to\infty} f'_k(x) \quad \forall x \in [a, b].$$

For completeness of description, we also show that functional sequence $f_k(x) = x^k$, considered in Problem 8.1, does not converge uniformly on the set $x \in [0,1]$.

For this purpose, let us first formulate in quantifier form *negation* definitions 8.4 and 8.5.

`Definition 8.6`	We will say that the functional sequence $\{f_k(x)\}$ *converges not uniformly on the set* X , if 1) $f_k(x) \underset{X}{\rightarrow} F(x)$ and 2) $\exists \varepsilon_0 > 0$ such that $\forall N \in \mathbb{N} : \exists x_0 \in X$ and $\exists n_0 \geq N$, for which inequality $$\Big| f_{n_0}(x_0) - F(x_0) \Big| \geq \varepsilon_0$$ is true.

Note that for the functional sequence $\{x^k\}$, on segment $[0,1]$

$$\exists\,\varepsilon_0 = \tfrac{1}{2} \quad \text{such that } \forall N: \quad \exists\, n_0 = N \text{ and } \exists\, x_0 = \frac{1}{\sqrt[n_0]{2}} < 1\,,$$

for which inquality

$$\Big|\, f_{n_0}(x_0) - F(x_0) \,\Big| = \Big|\, x_0^{n_0} - 0 \,\Big| = \left(\frac{1}{\sqrt[N]{2}}\right)^N = \frac{1}{2} \geq \varepsilon_0$$

is true. This, according to definition 8.6, proves the absence of uniform convergence.

Finally, taking into account the solution to Problem 8.1, we conclude that the considered functional sequence converges not uniformly on segment $[0,1]$.

Conditions for uniform convergence functional sequence

Let the members of functional sequence have such properties as continuity, integrability and differentiability.

In this case Theorems 8.1 — 8.3 state *sufficient* (but not necessary!) conditions under which there are the similar properties for the limit function.

Let us now formulate the criteria, guaranteeing the presence or absence of the property of uniform convergence for a functional sequence.

`Theorem 8.4` **In order for the functional sequence $\{f_k(x)\}$, defined on the set X , converged uniformly on this set to $F(x)$, necessary and sufficient to**

$$\lim_{k\to\infty}\sup_{x\in X}\Big|\, f_k(x)-F(x)\,\Big| = 0\,.$$

`Corollary 8.1` **If there is** *infinitesimal* **number sequence $\{A_k\}$ and number N such that**

$$\forall k \geq N \qquad \Big|\, f_k(x)-F(x)\,\Big| \leq A_k\,,$$

then $\quad f_k(x) \underset{X}{\rightrightarrows} F(x)\,.$

`Theorem 8.5 (Cauchy criterion)` **In order for the functional sequence $\{f_k(x)\}$, defined on the set X, converged uniformly on this set, necessary and sufficient to**

$\forall \varepsilon > 0 \quad \exists\, N_\varepsilon$ **such that**
$\forall k \geq N_\varepsilon\,, \quad \forall p \in \mathbb{N} \quad \forall x \in X,$

inequality $\left| \; f_{k+p}(x) - f_k(x) \; \right| < \varepsilon$ **is true.**

`Theorem 8.6 (negation of the Cauchy criterion)` **In order for the functional sequence $\{f_k(x)\}$, defined on the set X, did not converge uniformly on this set, necessary and sufficient to**

$\forall\; N \in \mathbb{N},$ **there are found**
$\varepsilon_0 > 0,\; k_0 \geq N,\; x_0 \in X$ **and** $p_0 \in \mathbb{N}$

such that inequality $\left| \; f_{k_0+p_0}(x_0) - f_{k_0}(x_0) \; \right| \geq \varepsilon_0$ **is true.**

The Cauchy criterion (as well as its negation) is convenient to use in cases where the limit function $F(x)$ is unknown or cannot be represented in a form convenient for use.

As an example, consider Problem 8.1 again and prove absence of uniform convergence for functional sequence $\{x^k\}$ on segment $[0,1]$.

Notice, that $\forall N \in \mathbb{N} \quad \exists\, n_0 = N \geq N$, $\exists p_0 = N$ and $\exists\, x_0 = \dfrac{1}{\sqrt[n_0]{2}} < 1$, for which the relations are valid

$$\left| \; f_{n_0+p_0}(x_0) - f_{n_0}(x_0) \; \right| = \left| \; x_0^{n_0+p_0} - x_0^{n_0} \; \right| =$$

$$= \left| \left(\frac{1}{\sqrt[N]{2}} \right)^{2N} - \left(\frac{1}{\sqrt[N]{2}} \right)^{N} \right| = \left| \frac{1}{4} - \frac{1}{2} \right| = \frac{1}{4} = \varepsilon_0 > 0\,.$$

This, by virtue of Theorem 8.6, proves the absence of uniform convergence.

Examples of studying functional sequences for convergence

Problem 8.2 *Investigate a functional sequence for uniform convergence*

$$f_k(x) = \frac{k \sin^2 kx}{x^2 + k^2} \qquad x \in [\,0, +\infty) \;.$$

Solution. 1) Let us first find $F(x)$ — the limit function. That is, for a fixed non-negative x we calculate

$$\lim_{k\to\infty} \frac{k \sin^2 kx}{x^2 + k^2} \,.$$

We have estimates

$$\left| \frac{k \sin^2 kx}{x^2 + k^2} \right| \le \frac{k}{x^2 + k^2} = \frac{1}{k + \dfrac{x^2}{k}} \quad \underset{k\to\infty}{\longrightarrow} \quad 0 \,.$$

It means $F(x) \equiv 0$.

2) Now on the *entire* set $[0,+\infty)$ we estimate

$$\left|\, f_k(x) - F(x) \,\right| = \left| \frac{k \sin^2 kx}{x^2+k^2} - 0 \right| \le \frac{k}{x^2+k^2} \le \frac{1}{k} .$$

Therefore, by virtue of Theorem 8.4, the sequence under study converges on the set $[0,+\infty)$ uniformly, because the

Solution found.

$$0 \;\le\; \lim_{k\to\infty} \sup_{x\in X} \left|\, f_k(x) - F(x) \,\right| \;\le\; \lim_{k\to\infty} \frac{1}{k} \;=\; 0\,.$$

Problem 8.3 *Investigate a functional sequence for uniform convergence*

$$f_k(x) = \frac{kx^2}{1+k^2x^4}\,.$$

on the: 1) $E_1 :\ x \in [0,1]$ and 2) $E_2 :\ x \in [1,+\infty)$.

Solution. 1) It is easy to see that $F(x) \equiv 0$ is a limit function on both sets.
We have

$$f'_k(x) = \frac{2kx(1-k^2x^4)}{(1+k^2x^4)^2} = 0$$

at the point $x_{0k} = \dfrac{1}{\sqrt{k}} \in E_1$, with $f_k(x_{0k}) = \dfrac{1}{2}$.
Whence it follows that all $f_k(x)$ monotonically decrease by E_2 and the estimates are valid on this entire set

$$\left|\, f_k(x) - F(x)\,\right| = \left|\, \frac{kx^2}{1+k^2x^4} - 0 \,\right| \le \frac{k}{1+k^2} < \frac{1}{k} \quad \underset{k\to\infty}{\longrightarrow} \quad 0\,.$$

This implies uniform convergence of the functional sequence under study on E_2.

2) Since $\forall k$ to E_1 there is a point $x_{0k} = \dfrac{1}{\sqrt{k}} \in E_1$, at which $f_k(x_{0k}) = \dfrac{1}{2}$, then

$$\frac{1}{2} \leq \sup_{x \in E_1} \Big|\, f_k(x) - F(x) \,\Big| \underset{k \to \infty}{\not\to} 0\,.$$

Solution found.

That is, the functional sequence under study does not converge uniformly on E_1, and converges unevenly on this set (see Fig. 2).

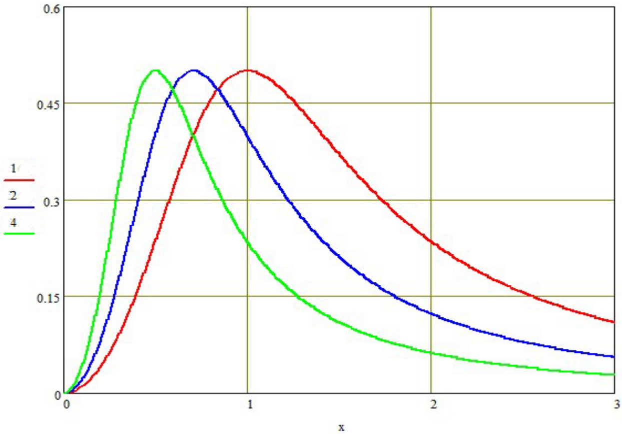

Fig. 2. Graphs of functions $f_k(x)$ for $k = 1, 2, 4$ to Problem 8.3.

Problem 8.4 *Investigate a functional sequence for uniform convergence*

$$f_k(x) = \frac{kx^2}{k^3 + x^3}\,.$$

on the: 1) $E_1 :\ x \in [0,1]$ *and* 2) $E_2 :\ x \in [1,+\infty)$.

Solution. 1) It is easy to see that $F(x) \equiv 0$ is a limit function on both sets.

In this case, the derivative $f_k'(x) = \dfrac{kx(2k^3 - x^3)}{(k^3 + x^3)^2} = 0$ at point $x_k^* = k\sqrt[3]{2} \in E_2\,,$ wherein $f_k(x_k^*) = \dfrac{\sqrt[3]{4}}{3}\quad \forall k \in \mathbb{N}\,.$

This means that all $f_k(x)$ increase monotonically by E_1 and the estimates are valid on this entire set

$$\left|\ f_k(x) - F(x)\ \right| = \left|\ \frac{kx^2}{k^3 + x^3} - 0\ \right| \le \frac{k}{1 + k^2} < \frac{1}{k} \quad \underset{k\to\infty}{\longrightarrow} \quad 0\,.$$

This implies the uniform convergence of the functional sequence under study on E_1.

2) Since for any k there is an extreme point on E_2 $x_k^* = k\sqrt[3]{2}$, wherein $f_k(x_k^*) = \dfrac{\sqrt[3]{4}}{3}$ $\forall k \in \mathbb{N}$, That

$$\frac{\sqrt[3]{4}}{3} \leq \sup_{x\in E_1} \Big| \, f_k(x) - F(x) \, \Big| \quad \underset{k\to\infty}{\not\to} \quad 0\,.$$

Solution found.

This means that the functional sequence under study does not converge uniformly on E_2, and converges unevenly on this set (see Fig. 3).

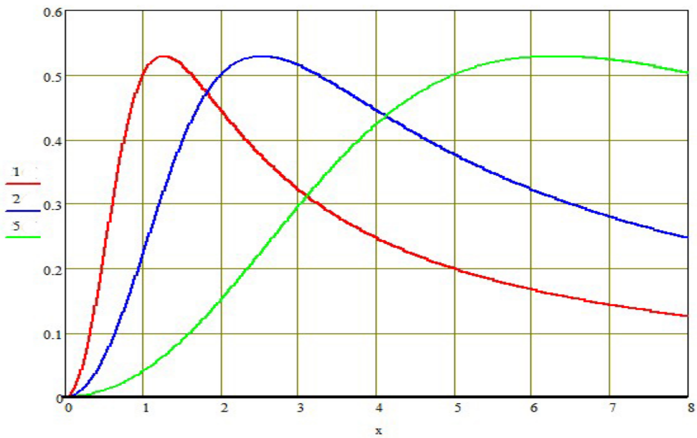

Fig. 3. Graphs of functions $f_k(x)$ for $k = 1, 2, 5$ to Problem 8.4.

To solve the next problem we need additional theoretical fact associated with the Taylor's formula.

It is known that if a function $f(x)$ has points in a neighborhood x_0 derivatives up to order $n-1$ inclusive, and at a point it has a derivative of order n, then the Taylor formula with remainder term *in Peano form* is valid

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o\big((x-x_0)^n\big)\,.$$

If we additionally require that the function $f(x)$ have in some neighborhood of the point x_0 derivatives up to order $n+1$ inclusive, then for any x from this neighborhood there is a point $\xi \in (x-x_0, x_0)\bigcup(x_0, x+x_0)$ such that the Taylor formula turns out to be valid with remainder *in Lagrange form*

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\,. \qquad (8.3)$$

This formula allows us to obtain useful estimates for function values.

For example, for the function $f(x) = \sin x$ for $n = 1$ due to (8.3) we have that

$$\exists \xi \in \mathbb{R}\,, \quad \text{for which} \quad \sin x = x + \frac{x^2}{2!}\left(\sin^{(2)} x\Big|_{x=\xi}\right).$$

Whence it follows that

$$\Big|\sin x - x\Big| \leq \frac{x^2}{2} \qquad \forall x \in \mathbb{R}\,.$$

`Problem 8.5` *Investigate a functional sequence for uniform convergence*

$$f_k(x) = k \operatorname{arctg} \frac{1}{kx} .$$

on the: 1) $E_1 : \ x \in (0,1)$ *and* 2) $E_2 : \ x \in (1,+\infty)$.

`Solution.` 1) For the functional sequence under study limit function for fixed x both sets have $F(x) = \dfrac{1}{x}$, since

$$\lim_{k\to\infty} k \operatorname{arctg} \frac{1}{kx} = \frac{1}{x} \left(\lim_{k\to\infty} kx \operatorname{arctg} \frac{1}{kx} \right) = \frac{1}{x} .$$

2) On the set E_1 for each number k there is a point $x_{0k} = \dfrac{1}{k}$, for which, due to $k \geq 1$, the following estimate holds:

$$\left| \, k \operatorname{arctg} \frac{1}{k x_{0k}} - \frac{1}{x_{0k}} \, \right| = \left| \, k \operatorname{arctg} 1 - k \, \right| \geq 1 - \frac{\pi}{4} > 0 \, .$$

This means that

$$\lim_{k \to \infty} \sup_{x \in X} \left| \, f_k(x) - F(x) \, \right| \neq 0$$

and the sequence under study does not converge uniformly on E_1.

3) Let us now consider the sequence under study on segment E_2 .
Function $\operatorname{arctg} x$ has a derivative of any order on the entire real axis. Therefore it will be fair for her the following version of Taylorp's formula with a remainder term in Lagrange form (8.3)

$$\operatorname{arctg} x = x + \frac{(\operatorname{arctg} x)''_{x=\xi}}{2!} x^2 .$$

Check for yourself what the value is $(\operatorname{arctg} x)''$ on the set E_2 does not exceed $1/2$. Therefore for this set the following estimate is valid: $\left|\operatorname{arctg} x - x\right| \leq \frac{x^2}{4}$, by virtue of which

$$\left| \; f_k(x) - F(x) \; \right| = \left| k \operatorname{arctg} \frac{1}{kx} - \frac{1}{x} \right| = k \left| \operatorname{arctg} \frac{1}{kx} - \frac{1}{kx} \right| \leq$$

$$\leq \quad k \, \frac{1}{4k^2x^2} \quad \leq \quad \frac{1}{4k} \quad \underset{k\to\infty}{\longrightarrow} \quad 0 \, .$$

Solution found.

Hence it converges uniformly.

To conclude the discussion of the properties of functional sequences, let's pay attention to the following important details.

1) The lack of uniform convergence may not be the reason for the presence of 'good' properties in the limit function. Let's consider

`Example 8.1.` Let (see Fig. 4)

$$f_k(x) = \begin{cases} k^2 x & \text{at} \quad x \in \left[0, \dfrac{1}{k}\right], \\ k^2 \left(\dfrac{2}{k} - x\right) & \text{at} \quad x \in \left(\dfrac{1}{k}, \dfrac{2}{k}\right], \\ 0 & \text{at} \quad x \in \left(\dfrac{2}{k}, 1\right]. \end{cases}$$

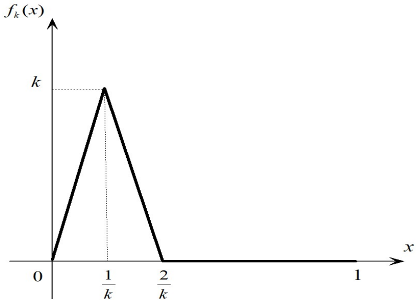

Fig. 4. Graph of the function $f_k(x)$ for Example 8.1.

In this example, the limit function $F(x) \equiv 0$. Indeed $\forall x^* \in (0,1]$ $\exists k$ such that $\dfrac{2}{k} < x^*$. And we have $f_k(0) = 0$ for each k.

All functions $f_k(x)$ $x[\in$ $[0,1]$ are continuous but not differentiable. Function sequence $\{f_k(x)\}$ is unlimited, which means that the convergence is non-uniform. Finally, the limit function has derivatives of any order.

2) The uniform convergence of a functional series with continuously differentiable terms does not guarantee the possibility of changing the order of operations differentiation and passage to the limit.

This fact illustrates

`Example 8.2.` Consider for $x \in [0,1]$ the functional sequence with a common member $f_k(x) = \dfrac{x^k}{k}$.

Due to the assessment $\left| \dfrac{x^k}{k} - 0 \right| \le \dfrac{1}{k}$ this sequence converges uniformly to a function. $F(x) \equiv 0$, for which $F'(x) \equiv 0$.

On the other side,

$$f_k'(x) = x^{k-1} \quad \Longrightarrow \quad f_k'(1) \to 1 \neq F(1) = 0\,.$$

Recall that condition for commutability of operations of passage to the limit and differentiation is given by Theorem 8.3.

3) We note that the method of replacing the functional sequence under study to an equivalent one when passing to the limit of the form $k \to \infty$ should be used with extreme caution since the discarded o-small terms may turn out to be converging unevenly. Let's consider

Example 8.3. Functional sequence $f_k(x) = \dfrac{1}{k} + \dfrac{x}{k^2}$ does not converge uniformly on $[0, \infty)$ to $F(x) \equiv 0$. Since from the existence of $x_0 = k^2$ follows (check it out yourself)

$$\lim_{k\to\infty} \sup_{x\in X} \Big| \, f_k(x) - F(x) \, \Big| \neq 0 \, .$$

At the same time, we have $\dfrac{x}{k^2} = o\left(\dfrac{1}{k}\right)$. Discarding this term will result in an error when determining the type of convergence. Since it is a fact that functional sequence $\phi_k(x) = \dfrac{1}{k}$ is uniformly convergent.