Number series

In some of applied problems, there is a need to use mathematical objects that are formally a sum of numbers with an unlimited number of terms.

Such objects, for historical reasons, are called $\mathit{number\ series}.$ They can be described using

Definition 7.1	Let some number sequence $\{a_n\}$ be given. Then, formally written, an expression of the form			
	$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$			
	is called a number series. The numbers $a_1, a_2, a_3, \ldots a_n, \ldots$ are members of a number series			

and

Definition 7.2	Sums of the first n terms of the number series $\sum_{k=1}^{\infty}a_k$
	$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^n a_k$
	are called <i>partial sums</i> of this series.

Finally

Definition	If exists $\lim_{n \to +\infty} S_n = A$, then the number series
7.3	$\sum_{k=1}^{\infty} a_k$ is called <i>convergent</i> , and the number A is called <i>the sum of the series</i> .
	Otherwise, the number series is called <i>divergent</i> .

Problem Find the sum of the series
$$\sum_{k=1}^{\infty} \frac{1}{k+k^2}$$
.

Solution. Note that in this case $a_k = \frac{1}{k+k^2} = \frac{1}{k} - \frac{1}{k+1}$. That's why

$$S_n = \sum_{k=1}^n \frac{1}{k+k^2} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$
$$= 1 - \frac{1}{n+1} \to 1 \quad \text{at} \quad n \to \infty .$$

Solution It means $\sum_{k=1}^{\infty} \frac{1}{k+k^2} = 1$.

Problem Find the sum of the series
$$\sum_{k=0}^{\infty} rac{1}{3^k}$$
.

Solution. In this problem

$$S_n = \sum_{k=0}^n \frac{1}{3^k} = \frac{1 - \frac{1}{3^{n+1}}}{1 - \frac{1}{3}} \to \frac{3}{2} \quad \text{at} \quad n \to \infty .$$

Solution That is, $\sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{3}{2}$.

Theorem If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=0}^{\infty} a_k$ diverges. 7.1

This is a necessary condition for the convergence of a number series. Illustrates it

Problem Examine the series for convergence
$$\sum_{k=1}^{\infty} \left(\frac{3k^3 - 1}{3k^3 + 2} \right)^{k^3}$$
7.3

Solution. We have

$$\lim_{k \to \infty} \left(\frac{3k^3 - 1}{3k^3 + 2} \right)^{k^3} = \lim_{k \to \infty} \left(1 - \frac{3}{3k^3 + 2} \right)^{k^3} =$$
$$= \lim_{k \to \infty} \left(1 - \frac{1}{k^3 + \frac{2}{3}} \right)^{k^3 + \frac{2}{3}} \cdot \lim_{k \to \infty} \left(1 - \frac{1}{k^3 + \frac{2}{3}} \right)^{-\frac{2}{3}} = \frac{1}{e} \neq 0.$$

. 3

Solution found.

Therefore, this series diverges.

Essentially Definition 7.3 is a specific case of the definition limit of a numerical sequence. However, its practical application limited by the presence of the number A in it, which is usually unknown.

Therefore, the convergence criteria who don't use A are very useful. An example of such a criterion can serve, say, Theorem 7.1. Another such criterion is

Theorem Number series series $\sum_{k=1}^{\infty} a_k$ converges *if and* 7.2 (Cauchy (Cauchy $\{S_n\}$ is *fundamental*, i.e.

> $\forall \varepsilon > 0 \ \exists N_{\varepsilon} \in \mathbb{N}$ such that for all natural $n \ge N_{\varepsilon}$ and $\forall p \in \mathbb{N}$ the inequality holds

$$\left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon \,.$$

This criterion is both *necessary* and *sufficient* condition for the convergence of a number series. Therefore, to prove the divergence of a number series can be used

Theorem Number series series $\sum_{k=1}^{\infty} a_k$ diverges *if and* 7.3 (negation only *if* of the $\exists \varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$ there are $n_0 \ge N$ and Cauchy $p_0 \in \mathbb{N}$, for which the inequality holds criterion)

$$\left|\sum_{k=n_0+1}^{n_0+p_0} a_k\right| \ge \varepsilon_0 \,.$$

The use of the Cauchy criterion and its negation illustrates

Problem Prove that:
7.4
1) row
$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$
 converges;
2) row $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Solution. 1) We have an assessment

$$\sum_{k=n+1}^{n+p} \frac{1}{k^2} \le \sum_{k=n+1}^{n+p} \frac{1}{(k-1)k} = \sum_{k=n+1}^{n+p} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}.$$

Whence it follows that

$$\begin{aligned} \forall \varepsilon > 0 \quad \exists N_{\varepsilon} = \left[\frac{1}{\varepsilon}\right] + 1 \quad : \qquad \forall n \ge N_{\varepsilon} \text{ and } \forall p \in \mathbb{N} \quad \longrightarrow \\ & \longrightarrow \quad \left|\sum_{k=n+1}^{n+p} \frac{1}{k^2}\right| < \varepsilon \,. \end{aligned}$$

Therefore, this number series converges.

2) Here we use a different estimate

$$\sum_{k=n_0+1}^{n_0+p_0} \left| \frac{1}{n_0+1} + \frac{1}{n_0+2} + \ldots + \frac{1}{n_0+p_0} \right| \ge \frac{p_0}{n_0+p_0}.$$

Since p_0 is any natural number, we take $p_0 = n_0$, and this gives $\frac{p_0}{n_0 + p_0} = \frac{1}{2}$.

As a result, we get that $\forall N \in \mathbb{N} \quad \exists \varepsilon_0 = \frac{1}{2} \quad : \qquad \exists n_0 = N + 1 \text{ and } p_0 = n_0 \quad \longrightarrow$ $\longrightarrow \quad \left| \sum_{k=n_0+1}^{n_0+p_0} \frac{1}{k} \right| \ge \frac{1}{2} = \varepsilon_0.$

Solution

found. Therefore, this number series diverges.

Let's consider another example of using the Cauchy criterion

Problem Prove that row
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$
 converges. 7.5

Solution. 1) We have

$$\left|\sum_{k=n+1}^{n+p} \frac{(-1)^k}{k+1}\right| = \\ = \left|\frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+4} - \frac{1}{n+5} + \dots + \frac{(-1)^{n+(p-1)}}{n+(p-1)} + \frac{(-1)^{n+(p-1)}}{n+p}\right| \le$$

(let's reduce the denominator of each positive term by 1)

$$\leq \left| \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+3} - \frac{1}{n+5} + \frac{1}{n+5} - \frac{1}{n+7} + \dots \right|$$

Note that if the last term has a positive sign, then it is destroyed with the previous one. Otherwise we simply discard it, increasing the estimate «from above».

As a result, we get that

$$\left|\sum_{k=n+1}^{n+p} \frac{(-1)^k}{k+1}\right| \le \frac{1}{n+1} \quad \forall p \in \mathbb{N}.$$

Carrying out reasoning as in 1) problem 7.4, we come to the conclusion that this number series converges according to the Cauchy criterion.

Let us pay attention to the following property of number series. The sum with an unlimited number of terms does not, generally speaking, have the property *associativity*.

We illustrate this fact with the following examples.

Problem	Show the inapplicability	$of \ the$	combining	property	for	a	sum
7.6	of the form						

 $A = 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + 1 + 2 - 3 + \dots$

Solution. 1) Let us first note that if all terms of a series are zeros, then it converges and its sum is zero. This is true because in this case the sequence of partial sums tends to zero.

2) Let's try to calculate the sum of the series under consideration, grouping the terms first as

$$A = (1+2-3) + (1+2-3) + (1+2-3) + (1+2-3) + (1+2-3) + \dots$$

In this case, we conclude that A = 0, since each sum in parentheses equals zero.

However, with another grouping method

$$A = 1 + (2 - 3 + 1) + (2 - 3$$

Solution we get A = 1. This means that the association rule is found. inapplicable for sums with an unlimited number of terms.

Problem 7.6 may not seem like a very convincing example, since the series considered in it is not convergent. Indeed, the *necessary* condition for the convergence of the series is not satisfied for it: $\lim_{k\to\infty} a_k \neq 0$. Therefore, consider again the number series from Problem 7.5.

Problem Show that the sum of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ depends on the way its members are grouped.

Solution. 1) The series converges (moreover, it can be shown that that it converges to ln 2). We have

$$A = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} =$$

= $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots + \frac{1}{2n-1} - \frac{1}{2n} + \dots$

Let us rearrange the terms of the series as follows

$$\begin{pmatrix} 1 - \frac{1}{2} - \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} - \frac{1}{6} - \frac{1}{8} \end{pmatrix} + \dots + \begin{pmatrix} \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} \end{pmatrix} + \dots \\ = \begin{pmatrix} \frac{1}{2} - \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{6} - \frac{1}{8} \end{pmatrix} + \dots + \begin{pmatrix} \frac{1}{4n-2} - \frac{1}{4n} \end{pmatrix} + \dots = \\ = \frac{1}{2} \begin{pmatrix} 1 - \frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{3} - \frac{1}{4} \end{pmatrix} + \dots + \frac{1}{2} \begin{pmatrix} \frac{1}{2n-1} - \frac{1}{2n} \end{pmatrix} + \dots = \\ = \frac{1}{2} \begin{pmatrix} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{2n-1} - \frac{1}{2n} + \dots \end{pmatrix} = \frac{A}{2}.$$

This means that changing the way terms are grouped in this case does not violate convergence, but changes the sum of the series.

Number series with non-negative terms

If $a_k \ge 0 \quad \forall k \in \mathbb{N}$, then for the number series $\sum_{k=1}^{\infty} a_k$ there are additional possibilities for studying it for convergence. Let's list the main ones.

TheoremA series with non-negative terms converges if and7.4only if when the sequence of its partial sums is
limited.

Sequence of partial sums of a number series with non-negative terms obviously monotonically increasing, since

$$\sum_{k=1}^{n} a_k = S_n \le S_{n+1} = \sum_{k=1}^{n+1} a_k = S_n + a_{n+1}.$$

A monotonically increasing numerical sequence bounded from above converges by Weierstrass's theorem.

Theorem
7.6 (2nd
comparisonNumber series
 $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge or diverge
f exist:riterionsimultaneously if exist:
and $0 \le b_k$,1) the number N is such that $\forall k \ge N : 0 < a_k$
and $0 \le b_k$,2) finite non-zero limit $\lim_{k \to \infty} \frac{b_k}{a_k}$.

To study the convergence of series with non-negative terms method *selecting the main part* can also be used.

Theorem If the function f(x) is non-negative and decreasing 7.7 (integral on $[1, +\infty)$, then the number series $\sum_{k=1}^{\infty} f(k)$ and criterion) improper integral $\int_{1}^{+\infty} f(x) dx$ converge or diverge simultaneously.

Problem Examine the series for convergence
$$\sum_{k=1}^{\infty} \frac{1}{k^{\lambda}}$$
.
7.8

Solution. From the integral feature it follows that this series converges and diverges simultaneously with the improper integral

$$\int_{1}^{+\infty} \frac{dx}{x^{\lambda}}.$$

Solution Therefore, the series under study converges for $\lambda > 1$ and found. diverges as $\lambda \leq 1$.

Finally, let us describe criteria Dalembert and Cauchy. They are quite important for practice.

 $\begin{array}{ll} \text{Theorem} & \text{If } a_k > 0 \ \forall k \in \mathbb{N} \text{ and exists } \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lambda, \text{ then series} \\ \text{7.9} \\ \text{(Dalembert } \sum\limits_{k=1}^{\infty} a_k \text{ converges for } \lambda < 1 \text{ and diverges for } \lambda > 1 \text{ .} \\ \text{criterion)} \\ \begin{array}{l} \prod\limits_{k=1}^{k=1} \text{ the case } \lambda = 1 \text{ the series can either converge or} \\ \text{diverge.} \end{array}$

 $\begin{array}{ll} \text{Theorem} & \quad \text{If } a_k \geq 0 \ \forall k \in \mathbb{N} \ \text{and exists } \lim_{k \to \infty} \sqrt[k]{a_k} = \lambda, \ \text{then series} \\ \text{7.10} & \quad \sum_{k=1}^{\infty} a_k \ \text{converges for } \lambda < 1 \ \text{and diverges for } \lambda > 1 \ \text{.} \\ \text{test}) & \quad \text{In the case } \lambda = 1 \ \text{the series can either converge or } \\ \text{diverge.} \end{array}$

Problem Examine series for convergence

7.9

t)
$$\sum_{k=1}^{\infty} \frac{k!}{k!}$$

$$\begin{array}{ll} 1) & \sum\limits_{k=1}^{\infty} \overline{k^k} ,\\ 2) & \sum\limits_{k=2}^{\infty} \frac{1}{(\ln k)^{\ln k}} .\end{array}$$

Solution. 1) Apply the 1st comparison criterion

$$\frac{k!}{k^k} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots k}{k \cdot k \cdot k \cdot k \dots k} \le \frac{1 \cdot 2 \cdot k \cdot k \dots k}{k \cdot k \cdot k \dots k} = \frac{1 \cdot 2}{k \cdot k} = \frac{2}{k^2}$$

Using the solution to problem 7.8, we find that the series converges.

2) Let's use the equality $\ln k = k^{\log_k (\ln k)} = k^{\frac{\ln \ln k}{\ln k}}$. Then the general term of the series will take the form $\frac{1}{k^{\ln \ln k}}$. Since for sufficiently large k it is true that $\ln \ln k > 2$, we get an estimate

$$\frac{1}{(\ln k)^{\ln k}} = \frac{1}{k^{\ln \ln k}} < \frac{1}{k^2}.$$

Solution So, according to the 1st criterion of comparison the series found.

Problem Examine series for convergence

7.10

1)
$$\sum_{k=1}^{\infty} \frac{k^k}{\lambda^k k!}, \quad \lambda > 0,$$

2)
$$\sum_{k=1}^{\infty} k^3 \left(\frac{2k+1}{3k+2}\right)^k.$$

Solution. 1) Let us apply Dalembert criterion, we have for $k \to \infty$

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1} \cdot \lambda^k \cdot k!}{\lambda^{k+1} \cdot (k+1)! \cdot k^k} = \frac{1}{\lambda} \left(1 + \frac{1}{k} \right)^k \longrightarrow \frac{e}{\lambda}.$$

we find that the series diverges for $\lambda < e$ and converges for $\lambda > e.$

2) Let us use the Cauchy test, we have for $k \to \infty$

$$\sqrt[k]{a_k} = \sqrt[k]{k^3 \left(\frac{2k+1}{3k+2}\right)^k} = \sqrt[k]{k^3} \frac{2k+1}{3k+2} =$$
$$= \left(\sqrt[k]{k}\right)^3 \frac{2k+1}{3k+2} = \longrightarrow \frac{2}{3} < 1.$$

This means that, according to Cauchy test, the series under consideration converges.

3) In conclusion let's try to use Cauchy test for the point 1). In other words, let's try to evaluate $\lim_{k\to\infty} \sqrt[k]{\frac{k^k}{\lambda^k k!}}$. Here you can apply the method of selecting the main part, using Stirling's formula $k! \sim \sqrt{2\pi k} \left(\frac{k}{e}\right)^k$, which gives the expected

$$\begin{array}{ll} \text{Solution} \\ \text{found.} \end{array} & \sqrt[k]{\frac{k^k}{\lambda^k k!}} = \frac{k}{\lambda \sqrt[k]{k!}} \sim \frac{k}{\lambda} \cdot \frac{e}{k} \sqrt{\frac{1}{\sqrt[k]{2\pi k}}} & \xrightarrow[k \to \infty]{} \frac{e}{\lambda} \, . \end{array}$$

To conclude the topic, we will give another example of using the method of selecting the main part.

Problem Examine the series for convergence 7.11

$$\sum_{k=1}^{\infty} \left(1 - \sqrt{\frac{k^2 - 1}{k^2 + 1}} \right)^{\lambda} \,.$$

Solution. 1) We apply Taylor's formulas for the binomial function twice, to select the main part of the expression in parentheses:

$$\begin{split} 1 - \sqrt{\frac{k^2 - 1}{k^2 + 1}} &= 1 - \sqrt{\frac{1 - \frac{1}{k^2}}{1 + \frac{1}{k^2}}} = 1 - \sqrt{\left(1 - \frac{1}{k^2}\right)\left(1 + \frac{1}{k^2}\right)^{-1}} = \\ &= 1 - \sqrt{1 - \frac{2}{k^2} + o\left(\frac{1}{k^2}\right)} = 1 - \left(1 - \frac{1}{k^2} + o\left(\frac{1}{k^2}\right)\right) \sim \frac{1}{k^2} \,. \end{split}$$

2) According to the 2nd criterion of comparison, the series under study will be converge and diverge at the same time

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\lambda}} \,.$$

Solution Using the solution to Problem 7.8 again, we find that the series under study converges at $2\lambda > 1 \implies \lambda > \frac{1}{2}$.

Alternating number series

Let's consider some types of number series, when the sign of a member of the number series may change when its number changes,

Absolutely convergent series

First let's give

Definition 7.4	Number	series	$\sum_{k=1}^{\infty} a_k$	is	called	absolutely
	converger	<i>it</i> , if the	series conv	erge	$s \sum_{k=1}^{\infty} $	$ a_k $.

The importance of the class of absolutely convergent series is, first of all, follows from the statement

Theorem Let the row $\sum_{k=1}^{\infty} a_k$ converges absolutely. Then 7.11 the series $\sum_{k=1}^{\infty} (a_k + b_k)$ converges (or diverges) simultaneously with the series $\sum_{k=1}^{\infty} b_k$. Let us indicate the main properties of absolutely convergent series.

Theorem Let the row $\sum_{k=1}^{\infty} |a_k|$ converges and its sum is equal 7.12 to S. Then the series $\sum_{k=1}^{\infty} a_k$ also converges. If it has the sum σ , then the estimate $|\sigma| \leq S$.

Theorem Let the row $\sum_{k=1}^{\infty} a_k$ converges absolutely and its sum is equal to S. Then the series obtained from $\sum_{k=1}^{\infty} a_k$ by arbitrary rearrangement of its terms, also converges absolutely and its sum is equal to S.

Note that rearranging terms in a number series gives a new series because a different number sequence generates a new series. However, under the conditions of Theorem 7.13, the sum of the series does not change.

Theorem If the rows $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge absolutely, 7.14 then for any λ and μ a series of the form

$$\sum_{k=1}^{\infty} \left(\lambda a_k + \mu b_k \right)$$

also converges absolutely.

Alternating series

Definition Alternating is called number series of the form
7.5
$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k \text{, in which } a_k \ge 0 \quad \forall k \in \mathbb{N}.$$

Let us describe some properties of alternating series.

Theorem If
$$\lim_{k\to\infty} a_k = 0$$
 And $\forall k \in \mathbb{N} : a_k \ge a_{k+1} \ge 0$,
7.15 (Leibniz then a series $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges.
test)

Theorem If all the conditions of Theorem 7.17 are satisfied 7.16 and $\sum_{k=1}^{\infty} (-1)^{k-1} a_k = S$, A $\sum_{k=1}^{n} (-1)^{k-1} a_k = S_n$, then the assessment is fair $\left| S - S_n \right| \le a_{n+1}$.

- Solution. In this example $a_k = \frac{1}{k^{\lambda}}$. This number sequence with positive terms and it will be infinitesimal, and monotonically decreasing, for $\lambda > 0$.

Monotonicity condition $a_k \ge a_{k+1}$ in Leibniz's criterion is essential, which illustrates

Problem Examine the series for convergence
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k} - (-1)^k}$$
.
7.13

Solution. Let's use the method of selecting the main part

$$\begin{aligned} \frac{(-1)^k}{\sqrt[3]{k} - (-1)^k} &= \frac{\frac{(-1)^k}{\sqrt[3]{k}}}{1 - \frac{(-1)^k}{\sqrt[3]{k}}} = \\ &= \frac{(-1)^k}{\sqrt[3]{k}} \left(1 + \frac{(-1)^k}{\sqrt[3]{k}} + \frac{1}{\sqrt[3]{k^2}} + o\left(\frac{1}{\sqrt[3]{k^2}}\right) \right) = \\ &= \frac{(-1)^k}{\sqrt[3]{k}} + \frac{1}{\sqrt[3]{k^2}} + \frac{(-1)^k}{k\sqrt[3]{k}} + o\left(\frac{1}{\sqrt[3]{k^2}}\right) \,. \end{aligned}$$

Then the series under study will take the form

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}} + \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} + \sum_{k=1}^{\infty} \left[\frac{(-1)^k}{k\sqrt[3]{k}} + o\left(\frac{1}{\sqrt[3]{k^2}}\right) \right] \,.$$

A series with a common term in square brackets converges absolutely. Therefore (by virtue of Theorem 7.11) the conditions for the convergence of the series under study will be the same as for the sum of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}} + \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}} + \sum$$

the first of which converges according to the Leibniz criterion, and the second diverges (see solution to Problem 7.8). This means that the series under study diverges, although the denominator of its common term is equivalent to $\sqrt[3]{k}$ at $k \to \infty$.

Solution found.

To study the convergence of series of a more complex structure you can try to use sufficient Dirichlet and Abel tests.

Theorem If the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are bounded, 7.17 (Dirichlet test) Theorem $\{b_k\}$ monotonic and infinitesimal, then a series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Theorem	If the sequence $\{b_k\}$ monotonous and limited and
7.18 (Abel's	a series $\sum_{k=1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

For a more accurate description of the conditions for the convergence of number series may also be useful

Definition 7.6	Number series $\sum_{k=1}^{\infty} a_k$ is called <i>conditionally</i>
	<i>convergent</i> if it converges, but at the same time the
	∞
	series $\sum a_k $ diverges.
	k=1

Problem Examine the series for convergence 7.14

$$\sum_{k=1}^{\infty} \frac{\cos \frac{\pi k}{4}}{(k+2)\sqrt{\ln^3(k+3)}} \, .$$

Solution. We use estimate

$$\frac{\cos\frac{\pi k}{4}}{(k+2)\sqrt{\ln^3(k+3)}} \le \frac{1}{k\ln^{3/2}k}.$$

Consequently, the series under study converges absolutely on the basis of comparison and integral criteria, since the improper integral of a function of constant sign converges $\int_{1}^{+\infty} \frac{dx}{x \ln^{3/2} x}.$

Solution found.

Problem For $\forall \lambda \in \mathbb{R}$ examine the series for convergence 7.15

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^{\lambda}}.$$

Solution. 1) For $\lambda \leq 0$ the series diverges, since

$$\lim_{k \to \infty} \frac{\sin k}{k^{\lambda}} \neq 0 \,,$$

that is, the necessary condition for the convergence of the number series is not satisfied.

2) For $\lambda > 1$ the series converges according to the comparison criterion, because the assessment is fair

$$\left| \frac{\sin k}{k^{\lambda}} \right| \le \frac{1}{k^{\lambda}},$$

3) For $0 < \lambda \leq 1$ the series diverges non-absolutely according to the Dirichlet criterion.

Indeed, let $a_k = \sin k$ and $b_k = \frac{1}{k^{\lambda}}$. In this case sequence $\{b_k\}$ is infinitesimal and monotonic, and a sequence of partial sums of the form (according to formulas known from trigonometry)

$$S_n = \sum_{k=1}^n a_k = \sum_{k=1}^n \sin k = \frac{\sin \frac{n}{2} \sin \frac{n+1}{2}}{\sin \frac{1}{2}} \le \frac{1}{\sin \frac{1}{2}}$$

limited.

4) For $0 < \lambda \leq 1$ the series does not converge at all by comparison due to estimates

$$\left| \frac{\sin k}{k^{\lambda}} \right| \ge \frac{\sin^2 k}{k^{\lambda}} = \frac{1}{2k^{\lambda}} + \frac{\cos 2k}{2k^{\lambda}},$$

since the series $\sum_{k=1}^{n} \frac{\cos 2k}{2k^{\lambda}}$ converges conditionally (see Solution found. above, item 3)), and a row $\sum_{k=1}^{n} \frac{1}{2k^{\lambda}}$ diverges.

In conclusion, let's look at examples of using the main part selection method, Leibniz's test and Abel's test.

Problem Examine the series for convergence 7.16

$$\sum_{k=1}^{\infty} (-1)^k \left(1 - \cos \frac{\pi}{\sqrt{k}} \right) \,.$$

Solution. Applying Taylor's formula, we get

$$(-1)^k \left(1 - \cos\frac{\pi}{\sqrt{k}}\right) = (-1)^k \frac{\pi^2}{k} - \left[\frac{1}{2}\frac{\pi^4}{k^2} + o\left(\frac{1}{k^3}\right)\right].$$

A series with a common term in square brackets converges absolutely, while the series $\sum_{k=1}^{\infty} (-1)^k \frac{\pi^2}{k}$ converges Solution conditionally according to Leibniz's criterion. This means that the original series also converges, but not absolutely. Problem Examine the series for convergence

7.17

$$\sum_{k=1}^{\infty} \frac{\cos k \cdot \operatorname{arctg} k}{\ln(k+1)}$$

Solution. Let's denote $a_k = \frac{\cos k}{\ln(k+1)}$ and $b_k = \operatorname{arctg} k$. Wherein row $\sum_{k=1}^{\infty} a_k$ converges according to the Dirichlet criterion, and the sequence $\{b_k\}$ — is monotonic and limited.

Solution Then the series under study converges according to Abel's criterion.

Problem Examine the series for convergence

7.18

$$\sum_{k=1}^{\infty} \left(\sqrt{k^2 + \frac{k}{2}} \cdot \ln\left(1 + \frac{1}{k}\right) \right)^{k^2} \, .$$

Solution. Let's try to apply the Cauchy test for a series of constant sign. That is, we need to evaluate $\lim_{k\to\infty} \sqrt[k]{a_k}$. We have

$$\sqrt[k]{a_k} = \left[\sqrt{k^2 + \frac{k}{2}} \cdot \ln\left(1 + \frac{1}{k}\right)\right]^k =$$

$$= \left[k\sqrt{1 + \frac{1}{2k}} \cdot \ln\left(1 + \frac{1}{k}\right)\right]^k =$$

$$= \left[k\left(1 + \frac{1}{4k} + o\left(\frac{1}{k}\right)\right) \cdot \left(\frac{1}{k} - \frac{1}{2k^2} + o\left(\frac{1}{k^2}\right)\right)\right]^k =$$

$$= \left[1 - \frac{1}{4k} + o\left(\frac{1}{k}\right)\right]^k \rightarrow e^{-\frac{1}{4}} < 1 \text{ at } k \rightarrow \infty.$$

This means that the series under study converges according to the Cauchy test.

Solution Pay attention to what was required when selecting the main found. part, number of terms in the expansion of a logarithm.