Continuity of functions of several variables

Definition 2.1	Function of several variables $f(x)$ is called *continuous* at the point $a \in R^n$, if the equality $\lim\limits_{x \to a} f(x) = f(a)$ is valid.

Note that although Definition 2.1 is the same in wording as determination of continuity for a function of one variable. Moreover, it is significantly more stringent, since the limit in this definition is *is the limit of a function of several variables.*

For functions of several variables we can use the concept of continuity in the sense of Definition 2.1. Such continuity is called *continuity over a set of* variables $\{x_1, x_2, \ldots, x_n\}$). In addition, we can also use the concept of continuity of the function $f(x)$ for a single variable. However, these concepts are not equivalent.

For example, the function

$$f(x,y) = \begin{cases} 0 & \text{at} \quad x^2+y^2=0\,, \\ \dfrac{xy}{x^2+y^2} & \text{at} \quad x^2+y^2 \neq 0 \end{cases}$$

is continuous at the origin with respect to each of its arguments. But in R^2 it is not continuous at this point in the sense of Definition 2.1.

Since the limit of a function of several variables can be considered not on the entire neighborhood of the point x_0, but only on some its subset.

At the same time, it is necessary to distinguish cases of continuity at the point x_0 in the sense of Definition 2.1 and continuity at this point over a given subset.

For example, the function

$$f(x,y) = \begin{cases} 0 & \text{at} \quad x^2+y^2=0\,, \\ y\,\sin\dfrac{1}{x} & \text{at} \quad x^2+y^2\neq 0 \end{cases}$$

is continuous at the origin over its domain of definition. However, it will not be continuous by Definition 2.1, since it is undefined *for any* non-zero values y where $x=0$.

Moreover, Some properties of functions of one variable also hold for functions of several variables. For example, there is

Theorem 2.1 **For functions $f(x)$ and $g(x)$ continuous at some point functions**

$$f(x) + g(x)\,,$$
$$f(x) \cdot g(x)\,,$$
$$\frac{f(x)}{g(x)} \quad \textbf{at} \quad g(x) \neq 0\,,$$
$$f\big(g(x)\big)$$

will also be continuous.

Problem 2.1 *Examine the function for continuity at the origin*

$$f(x,y) = \begin{cases} 0 & at \quad x^2+y^2=0\,, \\ \dfrac{x^2 y}{x^4+y^2} & at \quad x^2+y^2 \neq 0 \end{cases}$$

Solution. If you go to the origin along the positive semi-axis Oy, then on this trajectory $x(t) = 0$ and, therefore,

$$\lim_{t\to 0} \frac{0 \cdot y(t)}{0^4 + y^2(t)} = 0\,,$$

and, if you follow the trajectory

$$\begin{cases} x(t) = t\,, \\ y(t) = t^2\,, \end{cases}$$

then $\displaystyle \lim_{t\to 0} \frac{t^2 \cdot t^2}{t^4+t^4} = \frac{1}{2}$. This means that the limit does not exist and, therefore, there is no continuity at the beginning of the coordinates.

Solution found.

Problem 2.2 *Examine the function for continuity at the origin*

$$f(x,y)=\begin{cases} A & at \quad x^2+y^2=0\,, \\ \dfrac{x^2y^4}{x^4+y^2} & at \quad x^2+y^2\neq 0 \end{cases}$$

Solution. We have an assessment

$$\left|\frac{x^2y^4}{x^4+y^2}\right| = \left|\frac{x^2y}{x^4+y^2}\right|\,|\,y^3| \leq \frac{1}{2}|\,y^3|\,.$$

This means that on any trajectory leading to the origin,

$$\lim_{t\to 0}\frac{x^2(t)\cdot y^4(t)}{x^4(t)+y^2(t)}=0\,,$$

since on this trajectory $y(t)\to 0$.

This means that the limit of the function at the origin is equal to zero and, therefore, the continuity at the origin will be take place at $A=0$. For other values of A there is no continuity.

Solution found.

Let us present (for reference) some definitions related to the concept continuity of a function of several variables in R^n .

Definition 2.02	A function of several variables that is continuous at every point of some subset $\Omega \subseteq R^n$, is called *continuous on* Ω .

As with functions of one variable, the properties of functions of many variables are continuous on a set can differ significantly from the properties of the function, continuous at a point.

Let us give examples of such properties.

The following theorems hold.

Theorem 2.2 **Any function continuous on a closed and bounded set (compact), is bounded on this set and reaches its exact upper and lower bounds on it.**

Theorem 2.3 **If the function $f(x)$ is continuous in the domain $G \subseteq R^n$ and takes on two different meanings, then it takes in this area and any meaning contained between them.**

Differences in the properties of functions of several variables, continuous at a point and continuous at a set illustrates, for example, important concept *uniform continuity*, which describes

Definition 2.03	A function of several variables $f(x)$ is called *uniformly continuous on the set* $G \subseteq R^n$, If $$\forall\varepsilon > 0 \quad \exists\delta_\varepsilon > 0 \quad \text{such that}$$ $$\forall x', x" \in G, \quad \text{for which} \quad \rho\big(x', x"\big) < \delta_\varepsilon$$ $$\text{running} \qquad \big\vert\, f(x') - f(x")\,\big\vert < \varepsilon\,.$$

Note that determining that a function $f(x)$ *is not* uniformly continuous has the form

Negation of definition 2.03	The multivariable function $f(x)$ is not *uniformly continuous on the set* $G \subseteq R^n$, If $$\exists\varepsilon_0 > 0 \quad \text{such that} \quad \forall\delta > 0$$ $$\exists\, x_0', x_0'' \in G, \qquad \text{for which} \quad \rho\big(x_0', x_0''\big) < \delta$$ $$\text{and executed} \quad \big\vert\, f(x_0') - f(x_0'')\,\big\vert \geq \varepsilon_0\,.$$

Difference between the concepts of continuity and uniform continuity becomes more clear if we use the definition of limit functions of several variables according to Cauchy.

In this case, the continuity of $f(x)$ on the set G defines

Definition 2.04	A function of several variables that is continuous at every point of some subset $G \subseteq R^n$, is called *continuous on* G, If $\forall x' \in G$ and $\forall \varepsilon > 0 \quad \exists \delta_{x',\varepsilon} > 0$ such that $\forall x" \in G$ satisfying $0 < \rho(x', x") < \delta_{x',\varepsilon}$, performed $\big| f(x') - f(x") \big| < \varepsilon$.

Moreover, the definition of uniform continuity (obviously equivalent to definition 2.03) obtained from 2.04 with the following slight change to the latter:

Definition 2.05	A function of several variables that is continuous at every point of some subset $G \subseteq R^n$, is called *uniformly continuous on* G, If $\forall \varepsilon > 0 \quad \exists \delta_{\varepsilon} > 0$ such that $\forall x', x" \in G$ and, satisfying $0 < \rho(x', x") < \delta_{\varepsilon}$, performed $\big| f(x') - f(x") \big| < \varepsilon$.

Note that in Definition 2.05 the selection rule is δ_{ε} must be *the same* for all $x' \in G$, while Definition 2.04 allows, that this rule can be *different* at different points $x', a \in G$, as indicated by the double subscript in $\delta_{x',\varepsilon}$ of Definition 2.04.

When solving problems related to the use of the concept of uniform continuity The following statements may be helpful.

`Theorem 2.4 (Cantor)` **Function continuous on a compact set (on a segment as well) uniformly continuous.**

`Theorem 2.5` **Let the function** $f(x)$ **be differentiable on the interval** $[0,+\infty]$, **then**

1) **if** $f'(x)$ **is limited to** $[0,+\infty)$, **then** $f(x)$ **is uniformly continuous on** $[0,+\infty)$,

2) **if** $f'(x)$ **is infinitely large for** $x \to +\infty$, **then** $f(x)$ **is not uniformly continuous on** $[0,+\infty)$.

Problem 2.3 *Examine the function $f(x) = \sqrt{x}$ for uniform continuity on $[0, +\infty)$.*

Solution. Let's evaluate

$$\left| \sqrt{x_0 + \delta} - \sqrt{x_0} \right| = \frac{\delta}{\sqrt{x_0 + \delta} + \sqrt{x_0}}$$

Since $x_0 \in [0, +\infty)$, then

$$\frac{\delta}{\sqrt{x_0 + \delta} + \sqrt{x_0}} \quad \leq \quad \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} \,.$$

Choosing $\delta = \dfrac{\varepsilon^2}{2} < \varepsilon^2$, we finally get

$$\left| \sqrt{x_0 + \delta} - \sqrt{x_0} \right| \quad \leq \quad \sqrt{\delta} < \sqrt{\varepsilon^2} = \varepsilon$$

Solution found. So the function $f(x) = \sqrt{x}$ by definition 2.03 is uniformly continuous on the interval $[0, +\infty)$.

Problem 2.4 *Examine the function for uniform continuity* $f(x) = \dfrac{1}{x}$ *by* $(0,1)$.

Solution. Let's use the negation of Definition 2.03. For any $\delta > 0$ we take

$$\delta^* = \begin{cases} \delta, & \text{if} \quad \delta < \frac{1}{2}\,, \\ \frac{1}{2}, & \text{if} \quad \delta \ge \frac{1}{2} \end{cases}$$

Then for δ^* on $(0,1)$ you can always choose a pair of numbers $x_0' = \delta^*$ and $x_0'' = 2\delta^*$ such that What

$$\left| \frac{1}{x_0'} - \frac{1}{x_0''} \right| = \frac{|\, x_0' - x_0'' \,|}{x_0' x_0''} = \frac{\delta^*}{2\delta^{*2}} = \frac{1}{2\delta^*} \ge \varepsilon_0 = \frac{1}{2}.$$

So the function $f(x) = \dfrac{1}{x}$ according to the negation of Definition 2.03 is not uniformly continuous on the interval $(0,1)$.

Solution found.

Problem 2.5 *Examine the function for uniform continuity* $f(x) = \sqrt{x}\ln(1+x^2)$ *on* $[\,0,+\infty)$.

Solution. 1. Function $f(x) = \sqrt{x}\ln(1+x^2)$ is defined and continuous on $[\,0,+\infty)$. Its derivative

$$f'(x) = \frac{\ln(1+x^2)}{2\sqrt{x}} + \frac{2x\sqrt{x}}{1+x^2}$$

exists and is continuous $\forall x \in (\,0,+\infty)$.

2. Applying the Taylor formula, we obtain that

$$\lim_{x\to+0} f'(x) = 0\,,$$

and according to L'Hopital's formula we have

$$\lim_{x\to+\infty} f'(x) = 0\,,$$

3. Combining the results of points 1 and 2, we conclude that $f'(x)$ is bounded to $[\,0,+\infty)$. And, using statement 1) of Theorem 2.05, we come to the conclusion about the uniform continuity of the function $f(x)$ on $[\,0,+\infty)$.

Solution found.

Problem 2.6 *Examine the function for uniform continuity* $f(x) = xe^{\sin x}$ *on* $[\,0, +\infty\,)$.

Solution. 1. Note that although the derivative

$$f'(x) = e^{\sin x}\Big(1 + x\cos x\Big)$$

unlimited on $[\,0, +\infty\,)$, application of statement 2) of Theorem 2.05 is incorrect here, since this derivative is not infinitely large. Therefore, we will use the negation of Definition 2.03.

2. Select two points on the interval $[\,0, +\infty\,)$ $x_0' = 2\pi n$ and $x_0'' = 2\pi n + \dfrac{1}{n}$, where $n \in \mathbb{N}$. This is always possible.

Let's estimate the value

$$\big|\, f(x_0'') - f(x_0')\,\big| =$$

$$= \left(2\pi n + \frac{1}{n}\right) e^{\sin\left(2\pi n + \frac{1}{n}\right)} - 2\pi n\, e^{\sin 2\pi n} \geq 2\pi n \left(e^{\sin\frac{1}{n}} - 1\right) \geq$$

$$\geq 2\pi n \, \sin\frac{1}{n} \geq 2\pi n \left(\frac{1}{n} - \frac{1}{6n^3}\right) = 2\pi - \frac{\pi}{3n^2} > \pi\ .$$

3. So,

$$\exists\,\varepsilon_0 = \pi \qquad \text{such that} \qquad \forall\delta > 0 \qquad \exists n = \left[\frac{1}{\delta}\right] + 1,$$

at which points $\quad x_0' = 2\pi n; \qquad x_0'' = 2\pi n + \dfrac{1}{n}$

belong to $[\,0, +\infty)$ and ensure the fulfillment of the inequalities

$$\left|\, x_0'' - x_0' \,\right| = \frac{1}{n} < \delta \qquad \text{and} \qquad \left|\, f(x_0'') - f(x_0') \,\right| \geq \varepsilon_0\,.$$

`Solution found.` That is, due to the negation of definition 2.03, There is no uniform continuity for the function under consideration.