Power series

Functional series are used as a tool of approximating the functions under study.

In this case, it is first of all useful to pay attention to functions that are the simplest in form of representation. Use, for example, rows whose common terms are power algebraic monomials.

Let's give

Definition 10.1	Functional series of the form $$\sum_{k=0}^{\infty} c_k (z - z_0)^k \,, \qquad (10.1)$$ in which z_0 and c_k are given complex constants, and z is a complex independent variable, is called *power series.*

Recall that a complex number is usually written in standard form as $z = a + ib$. Here real numbers a and b are called respectively *real* and *imaginary* parts for complex number z. They are denoted as $a = \text{Re}z$ and $b = \text{Im}z$.

Real non-negative number $\sqrt{a^2 + b^2}$ is called the *modulus* of a complex number z and is denoted by $|z|$.

The set of numbers $\text{Re}z \ \ \forall a \in \mathbb{R}$ obviously coincides with the set of real numbers. In the special case when z is a real number, there is the equality $|z| = \sqrt{a^2} = |a|$. That is, the modulus of z coincides with the absolute value of the number a.

This allows us to consider some properties of series (10.1) also true for real power series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \,, \tag{10.2}$$

in which x_0 and a_k — given real constants, and x is a real independent variable.

Finally, in a large number of cases we can assume that $z_0 = 0$, and investigate, without loss of generality, power series of the form

$$\sum_{k=0}^{\infty} c_k z^k \,. \tag{10.3}$$

It is also natural to call the complex-valued series (10.1) *absolutely convergent* if the series converges pointwise $\sum\limits_{k=0}^{\infty} \left| c_k (z - z_0)^k \right| .$

Properties of power series

It is easy to see that any power series (10.1) is convergent for $z = z_0$. In this case there is a limit function, identically equal to zero.

Shape of convergence set for power series (10.3) can be obtained using the following theorem.

Theorem 10.1 (Abel's 1st theorem)	**If the power series (10.3) converges pointwise at some** $z^* \neq 0$, **then it** *pointwise and absolutely converges* **for any** z **such that** $\|z\| < \|z^*\|$, **and, if diverges at** $z^* \neq 0$, **then it** *diverges* **for any** z, **such that** $\|z\| > \|z^*\|$.

`Theorem 10.2` **For each power series (10.3) there is R (R is a non-negative number or $+\infty$) such that this series converges absolutely on the set $|z| < R$.**

`Definition 10.2`	Set $\{z : \quad |z| < R\}$ is called *circle of convergence*, and R is called *radius of convergence* of series (10.3) .

The convergence set of series (10.3) consists of its circle of convergence and, perhaps some boundary points of this circle.

`Theorem 10.3` **Let $A : \{z : \quad |z| \le r < R\}$ is a *closed* circle of radius r. Then power series (10.3) converges *absolutely and uniformly* in A.**

To estimate the radius of convergence of series (10.3) may be useful

Theorem 10.4

1) If there is a finite or infinite limit $\lim\limits_{k\to\infty}\left|\dfrac{c_k}{c_{k+1}}\right|$,

then $R=\lim\limits_{k\to\infty}\left|\dfrac{c_k}{c_{k+1}}\right|$.

2) If there is a finite or infinite limit $\lim\limits_{k\to\infty}\sqrt[k]{|c_k|}$,

then $\dfrac{1}{R}=\lim\limits_{k\to\infty}\sqrt[k]{|c_k|}$.

We also note that for real series (10.2) it is customary to use instead of *circle of convergence* the term *interval of convergence.*

Problem 10.1 *Find the convergence radius, convergence interval and examine for convergence at the ends of the convergence interval for power series*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \,.$$

Solution. 1) By Theorem 10.4 we have either

$$R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k\to\infty} \left| -\frac{k+1}{k} \right| = 1 \,,$$

or

$$\frac{1}{R} = \lim_{k\to\infty} \sqrt[k]{|c_k|} = \lim_{k\to\infty} \sqrt[k]{\left| \frac{(-1)^{k-1}}{k} \right|} = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{k}} = 1 \,.$$

It follows that $R_{hcx} = 1$, and the convergence interval $(-1, 1)$.

2) At the boundary of the convergence interval for $x = -1$ we have a number series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-1)^k = -\sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges according to the integral criterion.

At the other end, for $x = 1$, we have alternating number series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$, which converges according to Leibniz's criterion.

Solution found.

Differentiation and integration of power series

Let us consider the power series of the form (10.2)

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k\,, \tag{10.2}$$

in which x_0 and a_k are- given real constants, and h is real independent variable. It converges uniformly on any segment containing x_0 and owned *strict interior* of the convergence interval. Then we have the following theorem.

Theorem 10.5

1) On the convergence interval $(x_0 - R, x_0 + R)$ the limit function of the series (10.2) has derivatives of any order, which can be found by term-by-term differentiation of this series.

2) On the convergence interval $(x_0 - R, x_0 + R)$ the integral of the limit function of series (10.2) can be found term-by-term integration of this series.

3) During term-by-term differentiation or integration of the power series (10.2), the radius of convergence *does not change***.**

Term-by-term differentiation and integration allows you to find representations of new power series.

Let us look at examples based on using the formula for the sum terms of an infinitely decreasing geometric progression. We consider it as the limit function for power series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \qquad (10.4)$$

Note that the radius of convergence of this power series is $R = 1$.

It is easy to see that the term-by-term differentiation of equality (10.4) gives

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \qquad \longrightarrow \qquad \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \qquad (10.5)$$

and term-by-term integration

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\ln(1-x) \qquad \longrightarrow \qquad \sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\frac{1}{1-x}. \qquad (10.6)$$

If in the left equality (10.6) we replace x with $-x$, then for $x = 1$ we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k+1} = \ln(1+x) \qquad \longrightarrow \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \ln 2\,.$$

It is the sum of the number series we used in Problem 7.7.

Let us consider a less obvious example.

Problem 10.2 *Find the limit function and radius of convergence of the series*

$$\sum_{k=1}^{\infty} k^2 x^{k-1} .$$

Solution. 1) Let us introduce the notation $S(x) = \sum\limits_{k=1}^{\infty} k^2 x^{k-1}$. Then

$$\int S(x)\,dx = C_1 + \sum_{k=1}^{\infty} k x^k = C_1 + x \sum_{k=1}^{\infty} k x^{k-1} ,$$

where, in turn, we denote $Q(x) = \sum\limits_{k=1}^{\infty} k x^{k-1}$.

2) Now let's use the fact that

$$\int Q(x)\,dx = C_2 + \sum_{k=1}^{\infty} kx^k = C_2 + \frac{1}{1-x} - 1\,.$$

Note that we know the radius of convergence of this series. It is equal to 1 and does not change either when differentiating or integrating power series.

Differentiating the last equality we find that

$$Q(x) = \frac{d}{dx}\left(C_2 + \frac{1}{1-x} - 1\right) = \frac{1}{(1-x)^2}\,.$$

Finally we get that

$$S(x) = \frac{d}{dx}\left(C_1 + \frac{x}{(1-x)^2}\right) = \frac{1+x}{(1-x)^3}\,,$$

`Solution found.`

where $R_{cx} = 1$.

Taylor series

So far we have solved the problem of finding the limit function $f(x)$ for power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \,, \tag{10.2}$$

by known coefficients $a_k \ \forall k \in \mathbb{N}$ for this row.

Let us now consider the *inverse* problem: to construct a power series, having in the neighborhood of the point x_0 given limit function $f(x)$.

The condition for solving this problem gives the following theorem.

`Theorem 10.6` **Let the function $f(x)$ be the limit function for the power series (10.2) in some neighborhood of the point x_0.**
Then $f(x)$ has derivatives of any order in this neighborhood and the coefficients of series (10.2) are determined by the formulas

$$a_0 = f(x_0), \qquad a_k = \frac{f^{(k)}(x_0)}{k!} \quad \forall k \in \mathbb{N}\,.$$

Such a function $f(x)$ is called *regular* at x_0.

Definition 10.3	For a function $f(x)$ having at a point x_0 derivatives of any order, power series of the form $$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\,, \qquad (10.7)$$ is called the *Taylor series* of this function.

The converse of Theorem 10.6 is false. Not every infinitely differentiable function is a limit for its Taylor series. For example, this is the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{at} \quad x \neq 0, \\ 0 & \text{at} \quad x = 0, \end{cases} \qquad (10.8)$$

having at $x = 0$ zero derivatives of any order and, hence a Taylor series with zero limit function.

It can be shown that the function generating the Taylor series is the limit function for this series, if the function itself and *all* its derivatives *are simultaneously bounded* on the convergence interval.

The example with function (10.8) also shows that, despite the exterior similarity of *series* Taylor with *formula* Taylor, these methods approximations of a function in a neighborhood of a certain point are fundamentally different.

Indeed, function (10.8) can be represented by the Taylor formula as $f(x) = o((x - x_0)^n$ with a remainder term in Peano form. But (10.8) is not regular.

To prove regularity the existence of derivatives up to the n-th order inclusive is not enough. Here we need to prove that the n-th remainder of the power series converges to the zero function in some neighborhood of the point x_0.

Note that the nth remainder of the series can be represented or in integral form

$$r_n(x) = \frac{1}{n!}\int\limits_{x_0}^{x} (x-u)^n f^{(n+1)}(u)\,du\,,$$

or in Lagrange form

$$r_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{(n+1)},$$

where ξ belongs to the interval bounded by the points x_0 and x.

Representation of basic elementary functions by Taylor series

Let us consider formulas representing some elementary functions by Taylor series at $x_0 = 0$. Such series are usually called *Maclaurin series.*

To make it easier to memorize, we will divide these formulas into three groups.

1. *Exponential and trigonometric functions*:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,, \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\,, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}. \tag{10.9}$$

with $R_{cx} = +\infty$.

2. *Power function*:

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} C_\alpha^k x^k\,, \qquad \text{where} \quad C_\alpha^k = \frac{\alpha(\alpha-1)\ldots(\alpha-(k-1))}{k!}\,. \tag{10.10}$$

with $R_{cx} = 1$.

Important special cases: $\dfrac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k\,, \quad \dfrac{1}{1+x} = \sum\limits_{k=0}^{\infty} (-1)^k x^k.$

3. *Logarimic functions*:

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}\,, \qquad \ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}. \tag{10.11}$$

c $R_{cx} = 1$.

Examples of representation of functions by Taylor series or Maclaurin

An example of the effective use of power series expansion is

Problem 10.3 *Find the power series representation and radius of convergence this series for the function*

$$F(x) = \int\limits_0^x e^{-u^2}\, du\,.$$

`Solution.` 1) This integral is not written in elementary functions. However function $F'(x) = e^{-x^2}$ can be expanded using the first of formulas (10.9) into a power series of the form

$$F'(x) = e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!},$$

where $R_{cx} = +\infty$.

2) Integrating this equality, we get

$$F(x) = \int_0^x e^{-u^2}\, du = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)k!} \qquad (10.12)$$

`Solution found.` — formulaic representation for the «impossible» integral in the form of a power series with $R_{cx} = +\infty$.

Graphs of partial sums of series (10.12) are shown in Fig. 1. These graphs illustrate the fact that as n increases, the «quality» of approximation improves, and it depends on x : the smaller $|x|$, the better the quality.

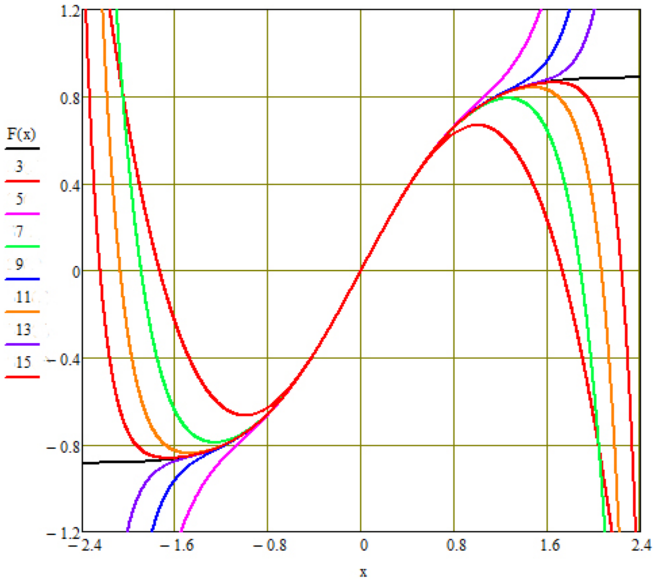

Fig. 1. Graphs of partial sums $S_n(x)$ series (10.12) of the function $F(x)$ for $2n+1 = 3, 5, 7, 9, 11, 13, 15$ in problem 10.3.

Let's consider a few more problems in which we need to represent a function sedately next.

Problem 10.4 *Expand a function*

$$f(x) = \frac{5-2x}{x^2-5x+6}$$

into the Maclaurin series and find the radius of convergence this row.

Solution. 1) Let's expand this function to the simplest fractions — I mean, convenient for using table rows

$$\frac{5-2x}{x^2-5x+6} = \frac{1}{2-x} + \frac{1}{3-x} = \frac{1}{2}\,\frac{1}{1-\frac{x}{2}} + \frac{1}{3}\,\frac{1}{1-\frac{x}{3}}\,.$$

2) The last formula from group (10.10) gives expansions

$$\frac{1}{1-\dfrac{x}{2}} = \sum_{k=0}^{\infty} \frac{x^k}{2^k} \qquad \text{and} \qquad \frac{1}{1-\dfrac{x}{3}} = \sum_{k=0}^{\infty} \frac{x^k}{3^k}\,,$$

the first of which has $R_{cx} = 2$, and the second one has $R_{cx} = 3$.

3) Using these expansions we find that

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x^k}{2^k} + \frac{1}{3}\sum_{k=0}^{\infty} \frac{x^k}{3^k} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{k+1}} + \frac{1}{3^{k+1}} \right) x^k\,,$$

`Solution received.`

for which it is obvious $R_{cx} = \min\{2, 3\} = 2$.

Problem 10.5 *Expand a function*

$$f(x) = \frac{6-3x}{\sqrt{x^2-4x+8}}$$

into a Taylor series in the vicinity of the point $x_0 = 2$ and find the radius of convergence of this series.

Solution. 1) In order to use table for the Maclaurin series (10.10) we first make a change of variable $u = x - 2 \implies x = u + 2$, what gives

$$f(x(u)) = \frac{6-3(u+2)}{\sqrt{(u+2)^2-4(u+2)+8}} = -\frac{3u}{\sqrt{4+u^2}} = .$$

$$= -\frac{3}{2}u\left(1+\left(\frac{u}{2}\right)^2\right)^{-\frac{1}{2}} .$$

2) Note that according to (10.10)

$$\left(1+\left(\frac{u}{2}\right)^2\right)^{-\frac{1}{2}} = 1+\sum_{k=1}^{\infty} C^k_{-\frac{1}{2}} \left(\frac{u}{2}\right)^{2k} \quad \text{c} \quad R_{cx} = 2\,,$$

where

$$C^k_{-\frac{1}{2}} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-(k-1)\right)}{k!} =$$

$$= \frac{(-1)^k\left(1\cdot 3\cdot 5\cdot \ldots (2k-1)\right)}{2^k k!} = \frac{(-1)^k\,(2k-1)!!}{2^k k!}\,.$$

3) Substituting, we get

$$f(x(u)) = -\frac{3}{2}u\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k\,(2k-1)!!}{2^k k!}\,\frac{u^{2k}}{2^{2k}}\right] = \,.$$

$$= -\frac{3}{2}u + 3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,(2k-1)!!}{2^{3k+1}k!}\,u^{2k+1}\,.$$

Returning to the original variable x gives the answer

$$f(x) = -\frac{3}{2}(x-2) + 3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,(2k-1)!!}{2^{3k+1}k!}\,(x-2)^{2k+1}\,,$$

Solution found.

where is the radius of convergence of the series $R_{cx} = 2\,.$

Problem 10.6 *Expand a function*

$$f(x) = x \operatorname{arctg}\left(x + \sqrt{1+x^2}\right)$$

Maclaurin series and find the radius of convergence of this series.

Solution. 1) In this task you can also use tabular expansions in the Maclaurin series (10.10). First let us find the Maclaurin series expansion for *derivative* of a function

$$\varphi(x) = \operatorname{arctg}\left(x + \sqrt{1+x^2}\right).$$

We have

$$\varphi'(x) = \frac{1}{1+(x+\sqrt{1+x^2})^2}\left(1+\frac{x}{\sqrt{1+x^2}}\right) =$$

$$= \frac{x+\sqrt{1+x^2}}{2\Big(1+x^2+x\sqrt{1+x^2}\Big)\sqrt{1+x^2}} = \frac{1}{2}\,\frac{1}{1+x^2}\,.$$

Here a question arises for the curious: the resulting formula is well known to us. Does this mean that $\varphi(x) = \frac{1}{2}\operatorname{arctg} x$?

2) Let us now expand $\varphi'(x)$ according to the Maclaurin formula, we get

$$\varphi'(x) = \frac{1}{2}\,(1+x^2)^{-1} = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k x^{2k}\,,$$

which after integration gives

$$\varphi(x) = C + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\,x^{2k+1}\,,$$

where $C = \frac{\pi}{4}$, since $\varphi(0) = \frac{\pi}{4}$, with $R_{cx} = 1$.

Now we write out the final answer

`Solution found.`

$$f(x) = x\varphi(x) = \frac{\pi}{4}x + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\,x^{2k+2}\,.$$

Problem 10.7 *Expand a function*

$$f(x)=\frac{1}{1+x+x^2}$$

Maclaurin series and find the radius of convergence of this series.

Solution. 1) It is easy to see that the value $x+x^2$ is small for small $|x|$. However, the formula

$$f(x)=\sum_{k=0}^{\infty}(x+x^2)^k$$

is not a solution to our problem. In the Maclaurin formula the powers of the variable x must be ordered in ascending order.

2) To overcome this difficulty, we transform this function in the following way

$$f(x)=\frac{1}{1+x+x^2}=\frac{1-x}{1-x^3}=(1-x)\sum_{k=0}^{\infty}x^{3k}=$$

$$=\sum_{k=0}^{\infty}x^{3k}-\sum_{k=0}^{\infty}x^{3k+1} \qquad \text{c} \;\; R_{cx}=1\,.$$

3) Write down the resulting series without using summation symbols

$$f(x)=1-x+0\cdot x^2+x^3-x^4+0\cdot x^5+x^6-x^7+\dots\,.$$

This series is generated by a numerical sequence

$$\{a_k\}=\Big\{\,1,\,-1,\,0,\,1,-1,\;0,\,1,-1,\,0,\,\dots\Big\},$$

which can also be given by the formula

$$\{a_k\}=\frac{2}{\sqrt{3}}\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}k\right) \qquad k=0,1,2,3,\dots\,.$$

This gives the answer

Solution found.

$$f(x)=\frac{2}{\sqrt{3}}\sum_{k=0}^{\infty}\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}k\right)x^k \qquad \text{c} \quad R_{cx}=1\,.$$

Elementary functions of a complex variable

Power series may be used to determine the exponential and trigonometric functions of a complex argument. In particular, by series converging in the entire complex plane, these functions are

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}\,, \tag{10.13}$$

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}\,, \tag{10.14}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}\,. \tag{10.15}$$

For all these series $R_{cx} = +\infty$, That is, they converge absolutely and uniformly $\forall z$ such as $|z| \le R < +\infty$.

Let us describe some properties of the series (10.13) — (10.15).

1. Using the theorem on the multiplication of absolutely convergent series, we can prove that

$$e^{z_1} e^{z_2} = e^{z_1+z_2} \qquad \forall z_1, z_2 \in \mathbb{C}\,. \tag{10.16}$$

2. Write down the coefficients of the series (10.13) — (10.15) for the functions e^{iz}, $\cos z$ and $\sin z$, (use according to the rule of multiplication of complex numbers $i^2 = -1$, $i^3 = -i$, $i^4 = 1, \ldots$) as a table

k	0	1	2	3	4	5	6	7	8	$\cdots$
e^{iz}	1	i	$-\frac{1}{2!}$	$-\frac{i}{3!}$	$\frac{1}{4!}$	$\frac{i}{5!}$	$-\frac{1}{6!}$	$-\frac{i}{7!}$	$\frac{1}{8!}$	$\cdots$
$\cos z$	1		$-\frac{1}{2!}$		$\frac{1}{4!}$		$-\frac{1}{6!}$		$\frac{1}{8!}$	$\cdots$
$\sin z$		1		$-\frac{1}{3!}$		$\frac{1}{5!}$		$-\frac{1}{7!}$		$\cdots$

It is easy to see that if the third line is multiplied by i and add with the second, you get the first line. Thus we arrive at *Euler's formula*

$$e^{iz} = \cos z + i \sin z\,. \tag{10.17}$$

3. From (10.14) it follows that $\cos z$ is an even function, and $\sin z$ is odd, so

$$e^{-iz} = \cos z - i\sin z\,. \tag{10.18}$$

Termwise addition of (10.17) and (10.18) gives

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \operatorname{ch} iz\,, \tag{10.19}$$

and term-by-term subtraction —

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \operatorname{sh} iz\,,$$

which partly justifies the use of the terms "hyperbolic sine"and «hyperbolic cosine».

Check for yourself that the function e^z *is periodic* with a period equal to $2\pi i$.

4. Shared use of equalities (10.16) and (10.17) allows *any* complex number $z = a + ib$ to be written as in *trigonometric* form

$$z = \rho\,(\cos\varphi + i\sin\varphi)\,,$$

and in *exponential*

$$z = \rho\, e^{i\varphi}\,,$$

where $\rho = \sqrt{a^2 + b^2} = |z|$, a φ on the interval $[0,\, 2\pi)$ uniquely determined by the system

$$\begin{cases} \cos\varphi = \dfrac{a}{\sqrt{a^2 + b^2}}\,, \\ \sin\varphi = \dfrac{b}{\sqrt{a^2 + b^2}}\,. \end{cases}$$

Convenience of using the exponential form of writing a complex number demonstrate the following tasks.

Problem 10.8 *Write the value of a complex number in standard form* $\sqrt[3]{-1}$.

Solution. 1) Recall that the standard form of a complex number is has the form $z = a + i\,b$.
To get it for a number $\sqrt[3]{-1}$, use sequentially standard, trigonometric (taking into account frequency) and exponential form complex number (-1):

$$(-1) = -1 + i \cdot 0 =$$

$$= \cos(\pi + 2\pi n) + i\,\sin(\pi + 2\pi n) = e^{i(\pi + 2\pi n)},$$

where $n \in \mathbb{Z}$.

2) In this problem, we need to take into account the periodicity of trigonometric functions is due to the fact that the symbol $\sqrt[3]{-1}$ can denote not *one*, but several complex numbers. Really,

$$\sqrt[3]{-1} = \sqrt[3]{e^{i(\pi+2\pi n)}} = e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}n\right)} =$$

$$= \cos\left(\frac{\pi}{3}+\frac{2\pi}{3}n\right) + i\sin\left(\frac{\pi}{3}+\frac{2\pi}{3}n\right) =$$

$$= \begin{cases} \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} & \text{at} \quad n=0\,, \\ \cos\pi + i\sin\pi = -1 + i\,0 & \text{at} \quad n=1\,, \\ \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} & \text{at} \quad n=2\,. \end{cases}$$

Solution found.

For other n values of the number $\sqrt[3]{-1}$ are repeated.

Problem 10.9 *Find the value of a complex number* i^i .

Solution. The number i is at the base of the power. Let's write it in exponential form as $e^{i\frac{\pi}{2}}$. Then, taking into account $i^2 = -1$, we get

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}} .$$

Solution found. This number is real.

Problem 10.10 *Find some real solution to the equation*

$$\cos\sqrt{x} = 5\,.$$

Solution. 1) According to formula (10.19) we have

$$\cos\sqrt{x} = \frac{e^{i\sqrt{x}} + e^{-i\sqrt{x}}}{2}\,.$$

Let us introduce a new unknown $u = e^{i\sqrt{x}}$. Then this equation will be the form

$$u + \frac{1}{u} = 10 \qquad \text{or} \qquad u^2 - 10u + 1 = 0\,.$$

Its solution is $u = 5 \pm \sqrt{24}\,.$

2) Now, for example, from the condition $e^{i\sqrt{x}} = 5 + \sqrt{24}$ we can find $x\,.$

$$i\sqrt{x} = \ln(5 + \sqrt{24}) \qquad \Longrightarrow \qquad x = -\ln^2(5 + \sqrt{24})\,.$$

Solution found. As an additional exercise, plot function $y = \cos\sqrt{x}$ on the entire real axis.