Space R^n

The term *function* denotes a three-component object, consisting of two *numerical* sets:

domain of definition X and *domain of values* Y,

as well as a *special rule* $f(...)$.

The rule matches

for *every* number from the domain of definition $x \in X$

a single number from a set of values $y \in Y$,

so that $y = f(x)$.

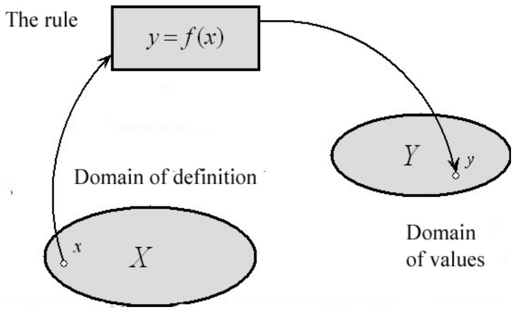

Fig.1-1. Function definition

It is also possible that the value of a function depends on several numeric arguments. In this case, we can introduce the concept functions *of several variables*. For example, like this:

> We will say that *a function of several variables* is given if some rule is specified. By this rule,
>
> > for every fixed set of ordered real numbers $\{x_1, x_2, \ldots, x_n\}$ from *domain of definition*
>
> a single number from *domain of values* is assigned.

We will denote a function of many variables as follows $f(x)$ or, in more detail, like $f(x_1, x_2, \ldots, x_n)$. Here x (without subscript) denotes the entire set $\{x_1, x_2, \ldots, x_n\}$.

Note that graphically this dependence depicted by the same figure. However, the points of the set X are no longer numbers.

This definition is similar to the definition of a function depending on one variable. It can be used just as effective if you overcome the following difficulty.

For a function of one variable $f(x)$ let's remember the definitions of *limit*, *continuity* and *derivative* of a function at a point.

They require evaluating the differences between the values of both a function and its argument. For this purpose, *difference modules* of numbers are used: $|\,x_{(1)} - x_{(2)}\,|$ and $|\,f(x_{(1)}) - f(x_{(2)})\,|$.

In the case of a function of many variables, the difference modulus is only suitable to estimate the difference between function values. And the question arises: how to quantify the closeness of two sets of ordered numbers, such as $\{x_{1(1)}, x_{2(1)}, \ldots, x_{n(1)}\}$ and $\{x_{1(2)}, x_{2(2)}, \ldots, x_{n(2)}\}$. [1]

[1] Let's agree that the subscript without parentheses is the number variable in the set, and the subscript in parentheses is the number of the variable set.

One possible way to quantify the proximity of such sets is as follows.

Consider the collection of *all possible* sets consisting of n ordered real numbers. We will call each such set *element* and write it as a string $\| \, x_1 \, x_2 \, \ldots \, x_n \, \|$. We will denote the entire set of elements as R^n.

Let us transform R^n into an n-dimensional *linear space*. To do this, we will use in R^n following rules:

— *equality of elements*:

$$\| \, x_1 \, x_2 \, \ldots \, x_n \, \|_{(1)} = \| \, x_1 \, x_2 \, \ldots \, x_n \, \|_{(2)} \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} x_{1(1)} = x_{1(2)} \, , \\ x_{2(1)} = x_{2(2)} \, , \\ \ldots \\ x_{n(1)} = x_{n(2)} \, . \end{array} \right.$$

— *summation of elements*:

$$\| \, x_1 \, x_2 \, \ldots \, x_n \, \|_{(1)} + \| \, x_1 \, x_2 \, \ldots \, x_n \, \|_{(2)} =$$

$$= \| \, x_{1(1)} + x_{1(2)} \quad x_{2(1)} + x_{2(2)} \quad \ldots \quad x_{n(1)} + x_{n(2)} \, \| \, .$$

— *multiplying a number by an element*:

$$\lambda \, \| \, x_1 \, x_2 \, \ldots \, x_n \, \| \; = \; \| \, \lambda \, x_1 \; \lambda \, x_2 \; \ldots \; \lambda \, x_n \, \| \, .$$

In a linear space there are no metric characteristics for its elements. For example, such as: *length, distance* or *angle value.*

However, they can be created if you turn R^n in *Euclidean space*. To do this, we introduce in R^n new operation *scalar product of elements* $x = \| x_1\, x_2, \ldots\, x_n \|$ and $y = \| y_1\, y_2, \ldots\, y_n \|$. This operation matches pair of elements x and y with a number denoted as (x, y) according to the formula

$$(x, y) = \sum_{k=1}^{n} x_k y_k \,.$$

Let, for example, in R^4 $x = \| 5,\, -2,\, 0,\, 3 \|$ and $y = \| -1,\, -1,\, 7,\, 2 \|$. Then

$$(x, y) = 5 \cdot (-1) + (-2) \cdot (-1) + 0 \cdot 7 + 3 \cdot 2 = 3\,.$$

The scalar product allows you to use for elements in R^n numerical (metric) characteristics such as:

— *norm* (*length*) of element x $\quad |x| = \sqrt{(x,x)} = \sqrt{\sum\limits_{k=1}^{n} x_k^2}\,,$

— *distance* between elements x and y

$$\rho(x,y) = |\,x-y\,| = \sqrt{(x-y,x-y)} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}\,. \quad (1.1)$$

Notice that element $x-y$ exists in R^n for *any pair* elements x and y. Indeed, in R^n (as in a linear space) element $x-y$ is the sum of x and $(-1)y$.

Moreover, in Euclidean space Euclidean space, for two *arbitrary* elements x and y the following relations are valid:

— *Cauchy-Bunyakovsky inequality* $|(x,y)| \leq |x|\,|y|$, which in R^n has the form

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \sqrt{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)}$$

— *triangle inequality* $|x+y| \leq |x| + |y|$, that in R^n there will be

$$\sqrt{\sum_{k=1}^{n} \left(x_k + y_k\right)^2} \leq \sqrt{\sum_{k=1}^{n} x_k^2} + \sqrt{\sum_{k=1}^{n} y_k^2}\ .$$

Let us present (for reference) some definitions of special elements and subsets in R^n .

Definition 1.1	The set of elements x such that $\rho(x, x_0) \le r$, called *ball* or *neighborhood* radius r centered at x_0 .

Definition 1.2	A neighborhood of the element x_0 is called *punctured*, if it consists of elements x_0, for which $0 < \rho(x, x_0) \le r$.

Definition 1.3	The element x_0 is called *interior* for the set $M \subset R^n$, if $x_0 \in M$ together with some ball of non-zero radius, centered at x_0.

Definition 1.4	The set M is called *open*, if all its elements are internal.

Definition 1.5	The element x_0 is called *limit point* of some set M if in any neighborhood of this element there is at least one element of M.

The limit point of a set M may or may not belong to M.

Definition 1.6	The set M is called *bounded*, if it is contained in some ball with non-zero radius.

Definition 1.7	The set M is called *closed*, if it contains all its limit points.

Definition 1.8	The element x is called *boundary*, if in any of its neighborhood exist as points belonging to M, and not belonging to this set.

Definition 1.9	They say that set of elements $$\|x(t)\| = \| \, x_1(t) \, x_2(t) \, \ldots \, x_n(t) \, \| \, ,$$ forms a *line* in R^n, if $x_k(t) \, \forall k = \overline{1,n}$ are continuous at $t \in [\alpha, \beta] \subset \mathbb{R}$ functions.

Definition 1.10	The set M is called *connected*, if any two of its points can be connected by a line, entirely owned by M.

Definition 1.11	An open and connected set is called a *domain*. A closed and bounded set is called *compact*.

Definition 1.12	*Level* of a function $f(x)$ is a collection of elements in R^n such that $f(x) = c \in \mathbb{R}$. In the case $n = 2$ we talk about *level line*, and in the case $n = 3$ the term *level surface* is used.

For small values of n it is customary to use index-free notation for the components of elements in R^n.

Example 1.1.1. Find for $z = f(x,y) = e^{2xy} - e^{xy} + 2$ domain of definition, domain of value and level lines

Solution. 1. In this case $n = 2$ and all operations used in the formula defining the function are satisfiable for any x and y, therefore the domain of definition there is an open set of the form $x \in (-\infty, +\infty)$ and $y \in (-\infty, +\infty)$.

2. Since the equality is true

$$z = f(x,y) = \left(e^{xy} - \frac{1}{2}\right)^2 + \frac{7}{4},$$

then the range of values will be the half-interval for which $z \in \left[\frac{7}{4}, +\infty\right)$.

3. From

$$\left(e^{xy} - \frac{1}{2}\right)^2 + \frac{7}{4} = const$$

we obviously have $xy = const$. Then the level lines for this function will be hyperbolas.

Limit of a function of several variables

In the space R^n we give

Definition 1.13	We will say that in R^n is given the sequence elements $\{x_{(k)}\}$, if for every natural number k is uniquely defined some element $\Vert\, x_{1(k)}\ x_{2(k)}\ \ldots\ x_{n(k)} \,\Vert \in R^n$.

Note that in this case each component of $\{x_{(k)}\}$, (that is, $\{x_{j(k)}\}\ \forall j = \overline{1, n}$,) is a *number* sequence in $\mathbb{R}$.

Now, using the metric $\rho(x, y)$ to estimate the proximity of x and y, we give

Definition 1.14	The element $a \in R^n$ is called the *limit* of the sequence $\{x_{(k)}\}$, If $\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ such that $\forall n \geq N_\varepsilon\ \ \rho(x_n, a) < \varepsilon$.

Symbolically it is denoted in R^n as $\lim\limits_{k\to\infty} x_{(k)} = a$.

As is easy to see, this definition repeats the definition of the limit of a number sequence for $\mathbb{R}$ in which $|\, x_n - a \,|$ is replaced by $\rho(x_n, a)$.

Now we can give the following definition of the limit of a function of several variables

Definition 1.15 (according to Heine)	The number A is called *limit of function* $f\big(x\big)$ *at the point* $a \in R^n$, If for *any* sequence $\{x_{(k)}\}$ such that $\lim\limits_{k\to\infty} x_{(k)} = a\,, \quad x_{(k)} \neq a\,,$ we have $\lim\limits_{k\to\infty} f\big(x_{(k)}\big) = A\,.$

And also, equivalent to definition 1.15,

Definition 1.16 (according to Cauchy)	The number A is called *limit of function* $f\big(x\big)$ *at the point* $a \in R^n$, If $\forall \varepsilon > 0 \;\; \exists\, \delta_\varepsilon > 0$ such that for all x, satisfying $0 < \rho(x, a) < \delta_\varepsilon\,,$ we have $\big\vert f(x) - A \big\vert < \varepsilon\,.$

Definitions 1.15 and 1.16 may be written as follows: $\lim\limits_{x\to a} f(x) = A\,,$ or in expanded form $\lim\limits_{\|x_1,\ldots,\,x_n\|\to\|a_1,\ldots,\,a_n\|} f\Big(x_1, \ldots,\, x_n\Big) = A\,.$

Limits of the form 1.15 (or 1.16) are called *limits at a point* (or, simply, *limits*).

In addition to them, for functions of several variables one can define *iterated limits*. Here the limit is taken sequentially for each of all variables, considering the rest as parameters.

For example, with $n = 2$ for the function $f(x, y)$ you can specify two iterated limits of the form

$$\lim_{y\to b}\big(\lim_{x\to a} f(x,y)\big) \qquad \text{and} \qquad \lim_{x\to a}\big(\lim_{y\to b} f(x,y)\big)\,.$$

For a function depending on n variables, the number of different iterated limits equals $n!$.

Note that in general the value of the iterated limit may be ambiguous. For example, for the function $u(x, y) = \dfrac{x-y}{x+y}$ we have

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right) = -1 \qquad \text{и} \qquad \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right) = 1\,.$$

While $\displaystyle\lim_{\|x,\,y\|\to\|0,\,0\|}\frac{x-y}{x+y}$ doesn't exist at all.

`Example 1.1.2.` Show that for the function

$$u(x,y) = \begin{cases} \dfrac{x^k y}{x^2+y^2}, & \text{if} \quad x^2+y^2 \neq 0\,, \\ 0\,, & \text{if} \quad x^2+y^2 = 0 \end{cases}$$

there is a limit at the origin for $k=2$ and does not exist for $k=1$.

`Solution.` 1. Let $k=2$. Apply definition 1.16 (according to Cauchy) and show that

$$\lim_{\|x,y\| \to \|0,0\|} \frac{x^2 y}{x^2+y^2} = 0\,.$$

Indeed, $\forall \varepsilon > 0$ we can take $\delta_\varepsilon = \varepsilon$, for which

$$|x| \leq \rho(x,y) = \sqrt{x^2+y^2} < \delta_\varepsilon = \varepsilon \qquad \forall y\,.$$

On the other hand, due to Cauchy's inequality,

$$\left| \frac{x^2 y}{x^2+y^2} - 0 \right| = |x| \frac{|x||y|}{x^2+y^2} \leq \frac{|x|}{2} < \varepsilon$$

on any trajectory leading to the origin.

2. Now we use another solution method. Let's move to the polar coordinate system, then from

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi \end{cases}$$

we have

$$\frac{x^2y}{x^2+y^2} = \frac{r^2\cos^2\varphi\, r\sin\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi} = r\cos^2\varphi\,\sin\varphi\,.$$

But this expression tends to zero, since on any trajectory leading to the origin of coordinates, that is, $\forall\varphi(r)$ holds $r\to 0$.

3. Let us now consider the case with $k = 1$. It's more convenient to use here definition (more precisely, the negation of definition) according to Heine.

Indeed, in order for the function $u(x,y)$ had no limit at the point $\| x_0\, y_0 \|$, is enough find two different sequences $\left\{ \| x_{(1)m} \;\; y_{(1)m} \| \right\}$ and $\left\{ \| x_{(2)m} \;\; y_{(2)m} \| \right\}$, $\|$ converging in R^2 as $m \to \infty$ to the limit point, on which number sequences

$$u\Big(x_{(1)m}\,,\; y_{(1)m}\Big) \qquad \text{and} \qquad u\Big(x_{(2)m}\,,\; y_{(2)m}\Big)$$

have different limits.

For Example 1.1.2, such sequences can be: converging to the origin,

sequences of the form $\left\{ \left\| \begin{matrix} \frac{1}{m} & 0 \end{matrix} \right\| \right\}$ and $\left\{ \left\| \begin{matrix} \frac{1}{m} & \frac{1}{m} \end{matrix} \right\| \right\}$.

For the first of them, we go to the origin along the Ox axis. For this sequence it will be

$$\lim_{m\to\infty} \frac{\frac{1}{m}\cdot 0}{\frac{1}{m^2}+0^2} = 0\,.$$

For the second, we go to the coordinate coordinates along the bisector of the first coordinate angle In this case we have

$$\lim_{m\to\infty} \frac{\frac{1}{m}\cdot\frac{1}{m}}{\frac{1}{m^2}+\frac{1}{m^2}} = \frac{1}{2}.$$

Therefore, according to the negation of Heine's definition of the limit, function specified in the formulation of Example 1.1.2, with $k = 1$ at the origin of the space R^2, has no limit.

The existence of a limit over an infinite number of trajectories does not imply the existence of a limit for all of them.

`Example 1.1.3.` Show that for the function

$$u(x, y) = x^2 e^{y-x^2}$$

1. There is a limit at $\| x\ y \| \to \|\infty \infty\|$ on the set T, of all rays emanating from the origin.
2. The limit $u(x, y)$ at $\| x\ y \| \to \|\infty \infty\|$ does not exist.

`Solution.` 1. The set T can be parametric given like this:

$$\begin{cases} x(t) = \alpha t\,, \\ y(t) = \beta t\,, \end{cases}$$

where $\alpha^2 + \beta^2 > 0$.

2. On a ray with fixed α and β choose a limit sequence of the form

$$\begin{cases} x_m = m\,\alpha\,, \\ y_m = m\,\beta\,. \end{cases}$$

Moreover, it is obvious that for $\alpha = 0$

$$\lim_{m\to\infty} u(x_m, y_m) = 0\,.$$

If $\alpha \neq 0$, then

$$\lim_{m\to\infty} u(x_m, y_m) = (m\alpha)^2 e^{m\beta-(m\alpha)^2} = 0\,,$$

which can be verified by applying, for example, L'Hopital's rule.

So, on the set T the limit of the function under consideration is 0.

3. Let us now show that there is no limit for the function under consideration. To do this, we choose a limit sequence of the form:

$$\begin{cases} x_m = m\,\alpha\,, \\ y_m = m^2\,\beta\,. \end{cases}$$

The points of this sequence lie on a parabola, and the exponent for each point is zero Then it is obvious that

$$\lim_{m\to\infty} u(x_m, y_m) = +\infty\,.$$

And the limit of the function in question does not exist, due to the negation of the definition of the limit according to Heine.