Quadratic forms in Euclidean space

Let us consider the problem of finding a basis in E^n which a quadratic form has a diagonal or standard form.

Recall: earlier we considered this problem in an arbitrary finite-dimensional space Λ^n, where it always had a solution, and, moreover, a non-unique one. In a finite-dimensional Euclidean space, as we will see, there are alternative, in many cases more effective methods for solving it.

As we know, any quadratic form in n-dimensional space is completely and uniquely described by a symmetric matrix and in an arbitrary basis $\{g_1, g_2, ..., g_n\}$ has the form

$$\Phi(x) = \sum_{k=1}^{n} \sum_{i=1}^{n} \varphi_{ki} \xi_k \xi_i = \| x \|_g^{\mathrm{T}} \| \Phi \|_g \| x \|_g .$$

The matrix of a quadratic form depends on the choice of basis and changes according to the following rule

$$\| \Phi \|_{g'} = \| S \|^{\mathrm{T}} \| \Phi \|_g \| S \| , \tag{1}$$

where $\| S \|$ is the transition matrix from the original basis to the new one $\{g'_1, g'_2, ..., g'_n\}$.

Now let the quadratic form $\Phi(x)$ be defined in the original orthonormal basis $\{e_1, e_2, ..., e_n\}$ of the Euclidean space E^n. Let us try to find in E^n another orthonormal basis $\{e_1, e_2, ..., e_n\}$ in which the form has a diagonal form.

Let us first note that in mathematical texts or statements, expressions such as "positive definite matrix", "eigenvectors of a matrix", etc. are often used (for the sake of brevity or because of contextual obviousness). These expressions are incorrect from a formal point of view, since the matrices are attributed properties that they do not possess.

Sign definiteness is a property of a quadratic form, while eigenvalues and eigenvectors are present in linear transformations. The reason for these natural "clauses" is that for both linear transformations and quadratic forms, the coordinate representations are square matrices.

Indeed, if we have some matrix, then by its appearance it is impossible to say whether this matrix is a record in E^n a linear transformation, a matrix of quadratic form, or a representation of some other object. For correctness, a more detailed description is required.

However, like everything in our world, it has a "reverse side of the coin", the noted ambiguity of terms can be used "for good".

We will assume that we have excellent observation and a wonderful memory. Thanks to which, we will first recall that

> self-conjugate linear transformations in E^n have an orthonormal basis consisting of its eigenvectors, in which the transformation matrix is diagonal.

On the other hand, the matrix of a quadratic form $\Phi(x)$ is symmetric and in the original orthonormal basis *can be considered* as the matrix of a self-conjugate transformation $\hat{\Phi}(x)$, which is usually called conjugate to the form $\Phi(x)$.

So, we have two objects different in nature: a quadratic form $\Phi(x)$ and an conjugate transformation $\hat{\Phi}(x)$, which have (by construction) the same matrix in the original orthonormal basis.

Now let's remember what happens to the matrices of these objects when replacing one orthonormal basis with another.

By virtue of (1), for a quadratic form we have

$$\|\Phi\|_{e'} = \|S\|^{\mathrm{T}} \|\Phi\|_{e} \|S\| \quad .$$

But for a linear transformation the rule of change is *different*:

$$\|\hat{\Phi}\|_{e'} = \|S\|^{-1} \|\hat{\Phi}\|_{e} \|S\|. \qquad (2)$$

In this situation, we can "help the grief" thanks to our excellent memory. We remember that the matrices of transition from one orthonormal basis to another orthonormal basis are (and only they!) orthonormal matrices.

And these matrices satisfy the equality $\|S\|^{-1} = \|S\|^{\mathrm{T}}$. But then the matrices are the same in the new orthonormal basis, since from (2) we have $\|\hat{\Phi}\|_{e'} = \|\Phi\|_{e'}$. The game is over!

Let us summarize our achievements in the form of a small but important generalization. Let us have in Λ^n the matrix of a quadratic form in the standard basis $\{g_1, g_2, ..., g_n\}$.

Let us transform (this is our right!) Λ^n into E^n , introducing a scalar product using the Gram matrix, which is the identity matrix. Then the basis $\{g_1, g_2, ..., g_n\}$ will become orthonormal and the symmetric matrix $\|\Phi\|_g$ can be taken as the matrix of the conjugate transformation.

Let us construct a basis from the eigenvectors of this transformation, in which its matrix will be diagonal (with eigenvalues $\hat{\Phi}(x)$ on the main diagonal). With this basis we remain in Λ^n, forgetting about E^n (this is, again, our right!)

It follows from our reasoning that

Theorem 1. **For every quadratic form defined in an orthonormal basis, there exists an orthonormal basis in which this form has a diagonal form.**

Let's consider the following example (which we solved earlier in Λ^3).

Task 9-01. *Using the orthogonal operator, reduce to diagonal form in* E^3 *quadratic form* $\Phi(x) = 2\xi_1\xi_2 + 2\xi_1\xi_3 - 2\xi_2\xi_3$.

Solution: 1°. Let the original ONB consist of elements $\{e_1, e_2, e_3\}$ with

$$\| e_1 \| = \left\| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\|, \| e_2 \| = \left\| \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\|, \| e_3 \| = \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\|.$$

We will restore its matrix from the quadratic form $\Phi(x) = 2\xi_1\xi_2 + 2\xi_1\xi_3 - 2\xi_2\xi_3$. We will obtain

$$\| \Phi \|_e = \left\| \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{matrix} \right\|.$$

2°. Let us consider the constructed symmetric matrix as defining a self-conjugate linear operator $\hat{\Phi}$ in E^3 and find its eigenvalues.

We compose and solve the characteristic:

$$\det\begin{Vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{Vmatrix} = 0 \qquad \text{or} \qquad \lambda^3 - 3\lambda + 2 = 0 \ .$$

It has roots: $\lambda_1 = -2,\ \lambda_{2,3} = 1$, which are the eigenvalues.

Note that if we are only interested in the diagonal form of the quadratic form, then we can write it now:

$$\Phi(x) = -2\xi_1'^2 + \xi_2'^2 + \xi_3'^2$$

and finish solving the problem.

3°. In the case where it is also required to find a diagonal basis for $\Phi(x)$, that is, to find a matrix $\|S\|$ – the transition matrix from the original ONB to the desired one, it is necessary to first find the eigenvectors of the operator $\hat{\Phi}$.

For $\lambda=-2$ we have $\begin{Vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{Vmatrix} \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}$, which gives $\begin{cases} 2\xi_1 + \xi_2 = -\xi_3, \\ \xi_1 + 2\xi_2 = \xi_3. \end{cases}$ Taking ξ_3 as a free unknown, we obtain the eigenvector $f_1 = \begin{Vmatrix} -1 \\ 1 \\ 1 \end{Vmatrix}$.

The multiplicity of the eigenvalue $\lambda=1$ is 2, which means that it must correspond to two linearly independent (but not necessarily orthogonal!) eigenvectors. The components of the eigenvector must satisfy the following system of equations:

$$\begin{Vmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{Vmatrix} \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix},$$

of which only one is independent $\xi_1 = \xi_2 + \xi_3$.

The general solution of this system will be of the form $\begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix} = \alpha \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix} + \beta \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} \quad \forall \alpha, \beta$. Each column of the form $\begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix}$ or $\begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix}$ is orthogonal , but they themselves are not orthogonal to each other.

Therefore, a pair of orthogonal eigenvectors corresponding to $\lambda = 1$, we form from the first fundamental solution and the orthogonal linear combination of the first and second.

The condition of orthogonality of columns $\begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix}$ and $\begin{Vmatrix} \alpha + \beta \\ \alpha \\ \beta \end{Vmatrix}$, is obviously $2\alpha + \beta = 0$.

Whence, for example, choosing and , we obtain $f_2 = \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix}$ and $f_3 = \begin{Vmatrix} -1 \\ 1 \\ -2 \end{Vmatrix}$.

The condition of orthogonality of columns and is obviously . Whence, for example, choosing and , we obtain and

4°. The set of elements $\{f_1, f_2, f_3\}$ is an orthogonal but non-normalized basis in E^3.

To construct an orthonormalized basis, we normalize each of the elements of the basis $\{f_1, f_2, f_3\}$. As a result, we obtain a matrix (orthogonal!)

$$\| S \| = \begin{Vmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{Vmatrix}$$

(transition from the "old" basis $\{e_1, e_2, e_3\}$ to the "new" basis $\{e'_1, e'_2, e'_3\}$), the columns of which are coordinate representations of the elements of the basis $\{e'_1, e'_2, e'_3\}$ by the basis $\{e_1, e_2, e_3\}$

We have already noted that this matrix is orthogonal (show it yourself!), that is, it satisfies the relation $\|S\|^{-1}=\|S\|^{\mathrm{T}}$. In turn, this allows us to easily obtain formulas expressing the "new" coordinates through the "old" ones.

Indeed, from the relation $\begin{Vmatrix}\xi_1\\ \xi_2\\ \xi_3\end{Vmatrix}=\|S\|\begin{Vmatrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{Vmatrix}$ it follows $\begin{Vmatrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{Vmatrix}=\|S\|^{-1}\begin{Vmatrix}\xi_1\\ \xi_2\\ \xi_3\end{Vmatrix}$, or finally

$\begin{Vmatrix}\xi'_1\\ \xi'_2\\ \xi'_3\end{Vmatrix}=\begin{Vmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{Vmatrix}\begin{Vmatrix}\xi_1\\ \xi_2\\ \xi_3\end{Vmatrix}$. This is the answer to the Task 9-01.

Solution is found

Construction of a basis in which two quadratic forms (one of which is of definite sign) have a diagonal form

Let a pair of quadratic forms and be given in some basis $\{g_1, g_2, ..., g_n\}$ of a linear space Λ^n, the first of which $\Phi(x)$ is of definite sign (for example, positive). Let us consider the problem of finding a basis $\{g'_1, g'_2, ..., g'_n\}$ in which the form $\{g'_1, g'_2, ..., g'_n\}$ has a standard form, and the form $\Psi(x)$ has a diagonal form.

We note in advance that the condition of sign-definiteness of one of the quadratic forms to be reduced to a diagonal form is essential, since in the general case two different quadratic forms may not be reduced to a diagonal form by a single linear change of coordinates

For example, a quadratic form $\Phi(x) = A\xi_1^2 + 2B\xi_1\xi_2 + C\xi_2^2$ in Λ^2 can be reduced to a diagonal form using a linear operator that reduces to a rotation of the plane of basis vectors by an angle α. In this case, it is necessary (check this, or remember the first semester and the theorem on reducing a second-order line to a standard form!), so that α satisfies the equation

$$(A - C)\sin 2\alpha = 2B\cos 2\alpha \ .$$

However, for a pair of quadratic forms

\

$$\Phi_1(x) = \xi_1^2 - \xi_2^2 \quad \text{and} \quad \Phi_2(x) = \xi_1\xi_2$$

an angle α that satisfies the system of conditions $\begin{cases} 2\sin 2\alpha = 0, \\ 0 = \cos 2\alpha, \end{cases}$ obviously does not exist.

Let us now describe an algorithm for reducing pairs of quadratic forms $\Phi(x)$ and $\Psi(x)$, defined in Λ^n some initial basis $\{g_1, g_2, ..., g_n\}$ (the first of which is positive definite), to standard and diagonal forms, respectively.

1°. Since the quadratic form $\Phi(x)$ is positive definite, there is another basis $\{g'_1, g'_2, ..., g'_n\}$ for it in Λ^n, in which it has a standard form, in which all coefficients are equal to one.

Let us reduce this form to this form by some method, for example, by selecting the perfect squares (Lagrange's method) with subsequent normalization of the elements of its matrix.

At the same time, we also transform the second quadratic form $\Psi(x)$ by the same method.

2°. Let us introduce into Λ^n the scalar product (standard) by the formula $(x, y) = \sum_{k=1}^{n} \xi'_k \eta'_k$, thereby transforming our linear space Λ^n into Euclidean E^n. Note that in this case the basis $\{g'_1, g'_2, ..., g'_n\} = \{e'_1, e'_2, ..., e'_n\}$ in which $\Phi(x)$ has a standard form is orthonormal.

3°. Now we construct a third, also orthonormal basis $\{e_1'', e_2'', ..., e_n''\}$, the transition to which is performed using a matrix $\|S\|$ according to the scheme described at the beginning of this text. In this third basis, the quadratic form $\Psi(x)$ is *diagonal.*

During this transition, the quadratic form $\Phi(x)$ will not lose its standard form, since it follows from the condition $\| \Phi \|_e = \| E \|$ and orthogonality $\|S\|$ that

$$\| \Phi \|_{e''} = \| S \|^{\mathrm{T}} \| \Phi \|_{e'} \| S \| = \| S \|^{\mathrm{T}} \| E \| \| S \| = \| S \|^{\mathrm{T}} \| S \| = \| S \|^{-1} \| S \| = \| E \|.$$

Thus, a basis is constructed in which the quadratic form $\Phi(x)$ has a standard form, and the form of $\Psi(x)$ is diagonal.

Finally, we note that the transition matrix from the original basis to the desired one is the product of

the *transition* matrix, in which the sign-definite quadratic form is reduced to standard form,

and

the *orthogonal* matrix $\|S\|$.

Let us demonstrate the use of the described approach using the following problem as an example.

Task 9-02. *Find a change of variables that brings the quadratic forms*

$$\Phi(x) = \xi_1^2 + 2\xi_1\xi_2 + 3\xi_2^2$$

and

$$\Psi(x) = 4\xi_1^2 + 16\xi_1\xi_2 + 6\xi_2^2$$

to standard and diagonal form, respectively.

Solution.

1°. We examine the quadratic forms $\Phi(x)$ and $\Psi(x)$ for sign definiteness. From the Sylvester criterion and the inequalities

$$\det\begin{Vmatrix} 1 & 1 \\ 1 & 3 \end{Vmatrix} = 2 > 0 \; ; \quad \det\begin{Vmatrix} 4 & 8 \\ 8 & 6 \end{Vmatrix} = -40 < 0$$

we conclude that $\Phi(x)$ is a positive definite quadratic form, while the form $\Psi(x)$ is not sign-definite.

2°. We bring the positive definite quadratic form $\Phi(x)$ to standard form using the Lagrange method. Since $\Phi(x) = \xi_1^2 + 2\xi_1\xi_2 + 3\xi_2^2 = (\xi_1 + \xi_2)^2 + 2\xi_2^2$, then, having made the change of variables

$$\begin{cases} \xi_1' = \xi_1 + \xi_2 \\ \xi_2' = \sqrt{2}\xi_2 \end{cases} \quad \text{or} \quad \begin{cases} \xi_1 = \xi_1' - \dfrac{1}{\sqrt{2}}\xi_2' \\ \xi_2 = \dfrac{1}{\sqrt{2}}\xi_2' \end{cases}$$

we obtain $\Phi(x) = \xi_1'^2 + \xi_2'^2$ and $\Psi(x) = 4\xi_1'^2 + 4\sqrt{2}\xi_1'\xi_2' - 3\xi_2'^2$, respectively.

3°. Introduction to Λ^2 the scalar product with the identity Gram matrix means that the coordinates $\{\xi_1';\xi_2'\}$ are the coordinates of the Euclidean space E^2 with the basis $\{e_1',e_2'\}$, where $\|e_1'\|_{e'} = \left\|\begin{matrix}1\\0\end{matrix}\right\|$; $\|e_2'\|_{e'} = \left\|\begin{matrix}0\\1\end{matrix}\right\|$. The matrix of the quadratic form $\Psi(x)$ in this basis is $\|\Psi\|_{e'} = \left\|\begin{matrix}4 & 2\sqrt{2}\\ 2\sqrt{2} & -3\end{matrix}\right\|$.

It defines the associated self-conjugate operator, which has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -4$, as well as orthonormal eigenvectors

$$\|f_1\|_{e'} = \left\|\begin{matrix}\frac{2\sqrt{2}}{3}\\ \\ \frac{1}{3}\end{matrix}\right\| \quad \text{and} \quad \|f_2\|_{e'} = \left\|\begin{matrix}-\frac{1}{3}\\ \\ \frac{2\sqrt{2}}{3}\end{matrix}\right\|,$$

which we will take as the desired, third basis $\{e_1'',e_2''\}$.

A graphical representation of the described procedure is shown in Fig. 1.

4°. The transition matrix from an orthonormal basis $\{e'_1, e'_2\}$ to an orthonormal basis $\{e''_1, e''_2\}$ in which $\Phi(x) = \xi_1''^2 + \xi_2''^2$ and $\Psi(x) = 5\xi_1''^2 - 4\xi_2''^2$, orthogonal and has the form $\|S\| = \left\| \begin{matrix} \frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2\sqrt{2}}{3} \end{matrix} \right\|$. Obviously, $\det\|S\| = 1$.

Whereby we finally obtain that

$$\begin{cases} \xi_1'' = \frac{2\sqrt{2}}{3}\xi_1' + \frac{1}{3}\xi_2' \\ \xi_2'' = -\frac{1}{3}\xi_1' + \frac{2\sqrt{2}}{3}\xi_2' \end{cases} \Rightarrow \begin{cases} \xi_1'' = \frac{2\sqrt{2}}{3}\xi_1 + \sqrt{2}\xi_2, \\ \xi_2'' = -\frac{1}{3}\xi_1 + \xi_2. \end{cases}$$

Solution is found

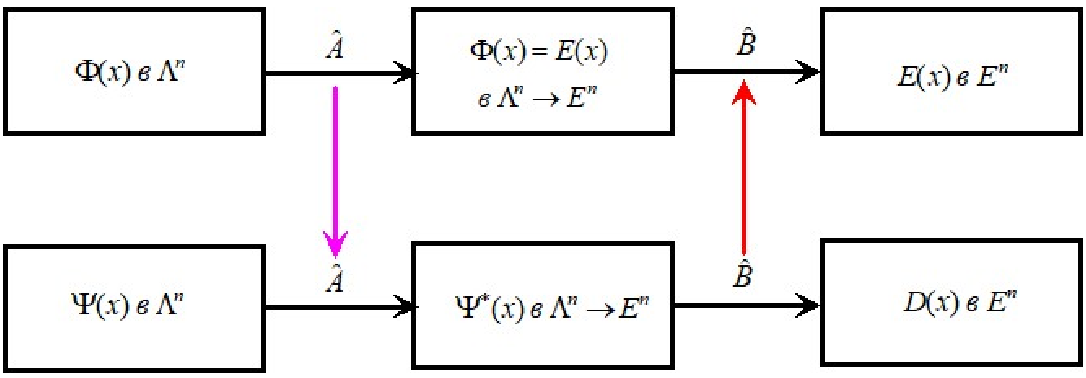

Fig. 1. Simultaneous diagonalization of a pair of quadratic forms.