Invariant subspaces of linear transformations

Definition

Let a linear operator $\hat{A}$, act in a vector space Λ and have values in the same space. In other words, the images and preimages for $\hat{A}$ belong to Λ, i.e. is a linear transformation of the space Λ. Then

> The set Ω is called an invariant subspace of the transformation $\hat{A}$,
> if $\forall x \in \Omega \rightarrow \hat{A}x \in \Omega$. (1)

Theorem. **The set of all eigenvectors corresponding to some $\lambda -$ eigenvalue of a linear transformation $\hat{A}$, supplemented by the zero-left element of the vector space Λ, is an *invariant* subspace $\hat{A}$.**

Proof.

Let $\hat{A}f_1 = \lambda f_1$ and $\hat{A}f_2 = \lambda f_2$. Then for any numbers α and β that are not simultaneously equal to zero:

$$\hat{A}(\alpha f_1 + \beta f_2) \quad = \quad \alpha \hat{A} f_1 + \beta \hat{A} f_2 \quad = \quad \alpha\lambda f_1 + \beta\lambda f_2 \quad = \quad \lambda(\alpha f_1 + \beta f_2),$$

which shows the validity of the theorem.

The theorem is proved.

Theorem **Any invariant eigenspace of a linear transformation $\hat{A}$ is also an *invariant* subspace of a linear transformation $\hat{B}$ if $\hat{A}$ and $\hat{B}$ *commute*.**

Proof.

Let Λ^* be an invariant eigenspace of $\hat{A}$, that is $\hat{A}f = \lambda f \quad \forall f \in \Lambda^*$. But then the equality $\hat{B}\hat{A}f = \hat{B}(\lambda f)$ holds, and due to the commutability and linearity of $\hat{A}$ and $\hat{B}$ will also hold $\hat{A}(\hat{B}f) = \lambda(\hat{B}f)$ for $\forall f \in \Lambda^*$.

The last condition means that $\hat{B}f \in \Lambda^*$ for $\forall f \in \Lambda^*$, that is, Λ^* is an invariant subspace of the operator $\hat{B}$.

The theorem is proved.

Task 5.01 *Find in Λ^3 all two-dimensional invariant subspaces of a linear transformation $\hat{A}$ for which in the standard basis*

$$\|\hat{A}\| = \begin{Vmatrix} -2 & -4 & 3 \\ 2 & 4 & -3 \\ 2 & 2 & -1 \end{Vmatrix}.$$

Solution. 1) For a linear transformation $\hat{A}$, the coordinates of the images are related to the coordinates of the preimages by the equality

$$\|x^*\| = \|\hat{A}\|\,\|x\|, \tag{2}$$

where $\|x\| = \|\ \xi_1\ \xi_2\ \xi_3\ \|^{\mathrm{T}}$ and $\|x^*\| = \|\ \xi_1^*\ \xi_2^*\ \xi_3^*\ \|^{\mathrm{T}}$.

Let the desired subspace be defined by the equation

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 = 0\,, \quad \text{where} \quad |\,\alpha_1\,| + |\,\alpha_2\,| + |\,\alpha_3\,| \neq 0\,.$$

The admissibility of such a description follows from the fact that a two-dimensional subspace in Λ^3 can be defined using a homogeneous system of linear equations with three unknowns, the rank of the main matrix of which is equal to one.

In this case, our problem is reduced to finding the values of the coefficients $\alpha_1, \alpha_2, \alpha_3$.

From the definition of invariance (1) it follows that the coordinates of the images of points of the desired subspace must satisfy the equation

$$\alpha_1\xi_1^* + \alpha_2\xi_2^* + \alpha_3\xi_3^* = 0\,. \tag{3}$$

2) Matrix equality (2) in coordinates takes the form of a system of equalities of the form:

$$\begin{cases} \xi_1^* = -2\xi_1 - 4\xi_2 + 3\xi_3, \\ \xi_2^* = \quad 2\xi_1 + 4\xi_2 - 3\xi_3, \\ \xi_3^* = \quad 2\xi_1 + 2\xi_2 - \ \xi_3. \end{cases} \tag{4}$$

Substituting (4) into (3), we obtain

$$\alpha_1(-2\xi_1 - 4\xi_2 + 3\xi_3) + \alpha_2(2\xi_1 + 4\xi_2 - 3\xi_3) + \alpha_3(2\xi_1 + 2\xi_2 - \ \xi_3) = 0 ,$$

and, after rearranging the terms,

$$\beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3 = 0 \ , \tag{5}$$

where

$$\begin{cases} \beta_1 = -2\alpha_1 + 2\alpha_2 + 2\alpha_3, \\ \beta_2 = -4\alpha_1 + 4\alpha_2 + 2\alpha_3, \\ \beta_3 = \quad 3\alpha_1 - 3\alpha_2 \ - \alpha_3. \end{cases} \tag{6}$$

Note that if (3) is an equation defining the sought subspace, then (5) will also be its description.

In this case, due to the homogeneity of equations (3) and (5), the coefficients in these equations can differ by the same constant factor. In other words,

$$\exists k: \qquad \beta_1 = k\alpha_1\,;\ \beta_2 = k\alpha_2\,;\ \beta_3 = k\alpha_3\,.$$

If these equalities are substituted into (6), then we obtain conditions defining the sought coefficients $\alpha_1, \alpha_2, \alpha_3$ of the following form:

$$\begin{cases} k\alpha_1 = -2\alpha_1 + 2\alpha_2 + 2\alpha_3, \\ k\alpha_2 = -4\alpha_1 + 4\alpha_2 + 2\alpha_3, \\ k\alpha_3 = \ \ 3\alpha_1 - 3\alpha_2 \ - \alpha_3 \end{cases} \quad \text{or} \quad \begin{cases} (-2-k)\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0, \\ -4\alpha_1 + (4-k)\alpha_2 + 2\alpha_3 = 0, \\ 3\alpha_1 - 3\alpha_2 + (-1-k)\alpha_3 = 0. \end{cases} \qquad (7)$$

Considering (7) as a homogeneous system of linear equations with unknowns $\alpha_1, \alpha_2, \alpha_3$, it is easy to see that the condition for the existence of non-zero solutions for (7) (see the theorem on the maximum number of linearly independent particular solutions of a homogeneous system of linear equations), which follows from the constraint

$$|\alpha_1| + |\alpha_2| + |\alpha_3| \neq 0 \ ,$$

has the form

$$\det \begin{Vmatrix} -k-2 & 2 & 2 \\ -4 & -k+4 & 2 \\ 3 & -3 & -k-1 \end{Vmatrix} = 0$$

or $k(k+1)(k-2) = 0$, i.e. non-zero solutions for system (7) can exist only for $k = 0, k = -1$ or $k = 2$.

3) By sequentially solving system (7) for these specific k, we obtain

$$\text{for } k=0 \quad \begin{cases} -2\alpha_1+2\alpha_2+2\alpha_3=0, \\ -4\alpha_1+4\alpha_2+2\alpha_3=0, \\ 3\alpha_1-3\alpha_2-\alpha_3=0. \end{cases} \Rightarrow \quad \begin{Vmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Vmatrix}=C\begin{Vmatrix}1\\ 1\\ 0\end{Vmatrix} \quad \forall C\neq 0;$$

$$\text{for } k=-1 \quad \begin{cases} -\alpha_1+2\alpha_2+2\alpha_3=0, \\ -4\alpha_1+5\alpha_2+2\alpha_3=0, \\ 3\alpha_1-3\alpha_2=0 \end{cases} \Rightarrow \quad \begin{Vmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Vmatrix}=C\begin{Vmatrix}-2\\ -2\\ 1\end{Vmatrix} \quad \forall C\neq 0;$$

$$\text{for } k=2 \quad \begin{cases} -4\alpha_1+2\alpha_2+2\alpha_3=0, \\ -4\alpha_1+2\alpha_2+2\alpha_3=0, \\ 3\alpha_1-3\alpha_2-3\alpha_3=0. \end{cases} \Rightarrow \quad \begin{Vmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Vmatrix}=C\begin{Vmatrix}0\\ -1\\ 1\end{Vmatrix} \quad \forall C\neq 0;$$

Using the found variants for the coefficients $\alpha_1, \alpha_2, \alpha_3$, we come to the conclusion that the linear transformation has three two-dimensional invariant subspaces, defined by the equations:

$$\xi_1+\xi_2=0\,; \qquad 2\xi_1+2\xi_2-\xi_3=0\,; \qquad \xi_2-\xi_3=0\,. \qquad (8)$$

4) We have obtained a solution to the problem, but upon careful analysis the following question arises.

If (3) is the equation of the image of the subspace

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 = 0\,,$$

then (7) is the equation of the preimage for this image.

What do the equalities $\beta_1 = \beta_2 = \beta_3 = 0$ mean in this case? After all, in this case it turns out that any point in Λ^3 has as its image a point in the subspace $\xi_1 + \xi_2 = 0$..

The fact is that the subspace $\xi_1 + \xi_2 = 0$ is the range of values of the transformation $\hat{A}$. (Check this yourself, using, for example, the Kronecker-Capelli theorem.)

This means that all points from the subspace $\xi_1 + \xi_2 = 0$ have their images in the same subspace. Therefore, it is invariant.

Going beyond the scope of the problem under consideration, we can also note that this transformation has invariant subspaces with a dimension different from 2.

First, the invariant subspaces for the transformation under consideration will formally be:

- the *zero* subspace, consisting only of the zero element (dimension 0);
- the entire space Λ^3 itself (dimension 3);
- each proper subspace (dimension 1).

Check for yourself the validity of these statements in the conditions of the problem under consideration.

In conclusion, let us pay attention to another, not immediately obvious, "inconsistency" in formulas (8).

For example, it may seem strange that the third invariant subspace $\xi_2 - \xi_3 = 0$ (by definition consisting of some values of the transformation) has an equation $\xi_1 + \xi_2 = 0$ that obviously does not coincide with the equation of the set of all values (images) of this transformation.

The point is that the coincidence of subspaces $\xi_2 - \xi_3 = 0$ and $\xi_1 + \xi_2 = 0$ in this problem is not required. It is sufficient that all images of points from belong to the set of values of $\hat{A}$, that is, satisfy $\xi_2 - \xi_3 = 0$ and $\xi_1 + \xi_2 = 0$.

Let us check that this is so. Let us find a formula for an arbitrary element of the invariant subspace $\xi_2 - \xi_3 = 0$. To do this, let us solve a system of homogeneous linear equations of the form

$$0 \cdot \xi_1 + \xi_2 - \xi_3 = 0 \ ,$$

taking ξ_2 as the main unknown , and as free unknowns ξ_1 and ξ_3. Then we obtain that

$$\left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| = C_1 \left\| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\| + C_2 \left\| \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\| \quad \forall C_1, C_2. \tag{9}$$

Note that the images of the basis elements of the obtained linear shell (9) will be equal to, respectively,

$$\begin{Vmatrix} -2 & -4 & 3 \\ 2 & 4 & -3 \\ 2 & 2 & -1 \end{Vmatrix} \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} = \begin{Vmatrix} -2 \\ 2 \\ 2 \end{Vmatrix} \qquad \text{and} \qquad \begin{Vmatrix} -2 & -4 & 3 \\ 2 & 4 & -3 \\ 2 & 2 & -1 \end{Vmatrix} \begin{Vmatrix} 0 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} -1 \\ 1 \\ 1 \end{Vmatrix} .$$

that is, they are nonzero, collinear to each other and belong to $\xi_1 + \xi_2 = 0$ - the subspace of the set of values of $\hat{A}$.

This means that the images of any points from the invariant subspace satisfy both the equation $\xi_2 - \xi_3 = 0$ and $\xi_1 + \xi_2 = 0$.

In other words, these images in their totality form a one-dimensional subspace simultaneously in two-dimensional subspaces $\xi_2 - \xi_3 = 0$ and $\xi_1 + \xi_2 = 0$.