

Calculating Limits Using Taylor's Formula

A representation of functions using Taylor's formula can be an effective method for calculating limits. More precisely, this representation can be used to expand uncertainties of the form $\frac{0}{0}$ and $\ll 1^\infty \gg$. The basis for this use is the following theorem (*theorem 6.1*).

Let functions $f(x)$ and $g(x)$ be such that $\lim_{x \rightarrow 0} f(x) = 0$, $\lim_{x \rightarrow 0} g(x) = 0$ and $\exists m \in \mathbb{N}$ such that the equalities $f(x) = ax^m + o(x^m)$ and $g(x) = bx^m + o(x^m)$ c $b \neq 0$. Then

$$1) \quad \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{ax^m + o(x^m)}{bx^m + o(x^m)} = \frac{a}{b},$$

$$2) \quad \lim_{x \rightarrow 0} \left(1 + f(x)\right)^{\frac{1}{g(x)}} =$$

$$= \lim_{x \rightarrow 0} \left(1 + ax^m + o(x^m)\right)^{\frac{1}{bx^m + o(x^m)}} = e^{\frac{a}{b}}.$$

Recall that the equality

$$y(x) = \sum_{k=0}^n \frac{y^{(k)}(x_0)}{k!} (x - x_0)^k + o((x - x_0)^n) \quad (6.1)$$

is called *the expansion of the function $y(x)$ in the neighborhood of the point x_0 by the Taylor formula with the remainder term in Peano form*. Equality (6.1) in the case when $x_0 = 0$ is called *the Maclaurin formula*.

We present a table of Maclaurin formulas for some basic elementary functions.

$$\begin{aligned}
 1) \quad e^x &= \sum_{k=0}^n \frac{x^k}{k!} + o(x^n) = \\
 &= 1 + x + \frac{x^2}{2!} + o(x^2), \\
 2) \quad \operatorname{sh} x &= \sum_{k=0}^n \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = \\
 &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6), \\
 3) \quad \operatorname{ch} x &= \sum_{k=0}^n \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) = \\
 &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5), \\
 4) \quad \sin x &= \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) = \\
 &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6), \\
 5) \quad \cos x &= \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) = \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5), \\
 6) \quad (1+x)^a &= 1 + \sum_{k=1}^n C_a^k x^k + o(x^n) = \\
 &= 1 + ax + \frac{a(a-1)}{2!} x^2 + o(x^2), \\
 7) \quad \frac{1}{1+x} &= \sum_{k=0}^n (-1)^k x^k + o(x^n) = \\
 &= 1 - x + x^2 + o(x^2), \\
 8) \quad \frac{1}{1-x} &= \sum_{k=0}^n x^k + o(x^n) = \\
 &= 1 + x + x^2 + o(x^2),
 \end{aligned}$$

$$\begin{aligned}
 9) \quad \frac{1}{\sqrt{1-x}} &= 1 + \sum_{k=1}^n \frac{(2k-1)!!}{2^k k!} x^k + o(x^n) = \\
 &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + o(x^2),
 \end{aligned}$$

$$\begin{aligned}
 10) \quad \ln(1+x) &= k = 1^n \frac{(-1)^{k-1}}{k} x^k + o(x^n) = \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3),
 \end{aligned}$$

$$\begin{aligned}
 11) \quad \operatorname{arctg} x &= \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2}) = \\
 &= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6).
 \end{aligned}$$

They may also be useful

$$12) \quad \operatorname{tg} x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6),$$

$$13) \quad \operatorname{arcsin} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + o(x^6).$$

Let us explain the idea of this method with the following example.

Example 6.1. Find $A = \lim_{x \rightarrow 0} \frac{\operatorname{sh} 2x - 2 \sin x}{x^3}$.

Solution. According to formulas 2) and 4), the following equalities will be true

$$\operatorname{sh} 2x = 2x - \frac{8x^3}{3!} + o(x^4) \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + o(x^4).$$

Therefore

$$\begin{aligned} A &= \lim_{x \rightarrow 0} \frac{2x + \frac{4}{3}x^3 + o(x^4) - 2x + \frac{1}{3}x^3 + o(x^4)}{x^3} = \\ &= \lim_{x \rightarrow 0} \frac{\frac{5}{3}x^3 + o(x^4)}{x^3} = \lim_{x \rightarrow 0} \left(\frac{5}{3} + \frac{o(x^4)}{x^3} \right) = \\ &= \frac{5}{3} + \lim_{x \rightarrow 0} \frac{o(x^4)}{x^3} = \frac{5}{3}. \end{aligned}$$

The above transformations do not require any special comments. Let us note only the equalities

$$\lim_{x \rightarrow 0} \frac{o(x^4)}{x^3} = \lim_{x \rightarrow 0} x \cdot \frac{o(x^4)}{x^4} = 0 \quad \text{and} \quad o(x^4) + o(x^4) = o(x^4).$$

The first of which follows from the definition of the function $o(x)$. In the second (it looks strange) all three functions $o(x)$ are *different*. A more precise notation would be $o_{(1)}(x^4) + o_{(2)}(x^4) = o_{(3)}(x^4)$.

Note that Theorem 6.1 does not provide a way to find the value of m . However, it is clear that this value is determined by the first nonzero terms in the expansions for the functions $f(x)$ and $g(x)$.

When solving problems, one can only recommend starting the construction of the expansion with the simplest of these two. The following examples illustrate the expediency of this approach.

Example 6.2. Find $A = \lim_{x \rightarrow 0} \frac{x\sqrt{1 + \sin x} + \ln(1 - x)}{\operatorname{tg} x - \sin x}$.

Solution. 1) The denominator here is clearly simpler than the numerator. Therefore, we will start with the first. From table formulas 4) and 12) we have

$$\operatorname{tg} x = x + \frac{x^3}{3} + o(x^4) \quad \text{and} \quad \sin x = x - \frac{x^3}{6} + o(x^4).$$

Therefore, the denominator is $\frac{1}{2}x^3 + o(x^4) \rightarrow m = 3$.

2) In the numerator, the first term is equal to the product of x and a complex (i.e. superposition) function $\sqrt{1 + \sin x}$. Therefore, it is sufficient to expand the latter to $o(x^2)$, since $x o(x^2) = o(x^3)$.

By tabular formula 4), where $a = \frac{1}{2}$

$$\begin{aligned}\sqrt{1+t} &= 1 + \frac{1}{2}t + \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right)}{2!}t^2 + o(t^2) = \\ &= 1 + \frac{1}{2}t - \frac{1}{8}t^2 + o(t^2).\end{aligned}$$

Therefore

$$\begin{aligned}\sqrt{1 + \sin x} &= 1 + \frac{1}{2} \left(x - \frac{1}{6}x^3 + o(x^4) \right) - \\ &\quad \frac{1}{8} \left(x - \frac{1}{6}x^3 + o(x^4) \right)^2 + o(x^2) = \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2).\end{aligned}$$

Note that here we used the well-known formula

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc. \quad (6.2)$$

Then we collected into $o(x^2)$ all terms of the second or higher order of smallness.

3) The table formula 10) for the logarithmic function yields

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + o(x^3),$$

which, together with the expansion obtained in 2), allows to write the Maclaurin expansion for the entire numerator

$$\begin{aligned} x \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2) \right) - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + o(x^3) &= \\ &= -\frac{11}{24}x^3 + o(x^3). \end{aligned}$$

$$\text{And finally, } A = \lim_{x \rightarrow 0} \frac{-\frac{11}{24}x^3 + o(x^3)}{\frac{1}{2}x^3 + o(x^3)} = -\frac{11}{12}.$$

Example 6.3. Find $A = \lim_{x \rightarrow 0} \frac{e^x + \ln(1 - \sin x) - 1}{\sqrt[3]{8 - x^4} - 2}$.

Solution. 1) The denominator in this problem is also simpler than the numerator. For the denominator, using table formula

6) with $a = \frac{1}{3}$, we have

$$\begin{aligned} \sqrt[3]{8 - x^4} - 2 &= 2 \left(\sqrt[3]{1 - \frac{1}{8}x^4} - 1 \right) = \\ &= 2 \left(1 - \frac{1}{3} \cdot \frac{x^4}{8} + o(x^4) - 1 \right) = -\frac{1}{12}x^4 + o(x^4). \end{aligned}$$

So, $m = 4$.

2) To construct the expansion of the numerator, we need the following table formulas

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4),$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + o(x^4),$$

$$\sin x = x - \frac{x^3}{6} + o(x^4).$$

Specifically, for the second term in the numerator, again using formula (6.2), we find

$$\begin{aligned} \ln(1-\sin x) &= \\ &= -\left(x - \frac{x^3}{6} + o(x^4)\right) - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 - \\ &\quad - \frac{1}{3}\left(x - \frac{x^3}{6} + o(x^4)\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{6} + o(x^4)\right)^4 + o(x^4) = \\ &= -x + \frac{x^3}{6} - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^3}{3} - \frac{x^4}{4} + o(x^4) = \\ &= -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + o(x^4). \end{aligned}$$

As a result, the formula for the numerator takes the form

$$\begin{aligned}
 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}-x-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{12}-1+o(x^4) = \\
 = -\frac{x^4}{24}+o(x^4).
 \end{aligned}$$

3) Thus, finally, we get

$$A = \lim_{x \rightarrow 0} \frac{-\frac{1}{24}x^4 + o(x^4)}{-\frac{1}{12}x^4 + o(x^4)} = \frac{1}{2}.$$

Sometimes, to construct the Maclaurin approximation, it is expedient to combine the use of tabular expansions and its definition, i.e. formula (6.1). Let us give an example of such a situation.

Example 6.4. Find $A = \lim_{x \rightarrow 0} \frac{\operatorname{arctg}(3 + x^2) - \operatorname{arctg}(2 + \cos x)}{\ln(1 + x) - e^x + 1}$.

Solution. 1) Since the expansions are valid

$$e^x = 1 + x + \frac{x^2}{2!} + o(x^2) \quad \ln(1 + x) = x - \frac{x^2}{2} + o(x^2),$$

then for the denominator we easily obtain

$$x - \frac{x^2}{2} + o(x^2) - 1 - x - \frac{x^2}{2!} + o(x^2) = -x^2 + o(x^2).$$

- 2) In this example $m = 2$ and the expansions for the numerator are easy to calculate using formula (6.1). To do this, at the point $x = 0$ we need to find the values of both the numerator itself and its derivatives up to and including the second order.

The value of the numerator $\Phi(x)$ at zero is obviously zero. Let us write out the formula for its first derivative, without making any simplifications. We have

$$\Phi'(x) = \frac{2x}{1 + (3 + x^2)^2} - \frac{-\sin x}{1 + (2 + \cos x)^2}$$

It is also obvious that $\Phi'(0) = 0$.

3) $\Phi''(x)$ is sought by the differentiation rule. We have for the first term in $\Phi'(x)$

$$\begin{aligned} \left(\frac{2x}{1 + (3 + x^2)^2} \right)'_x &= \frac{2(1 + (3 + x^2)^2) - 2x(1 + (3 + x^2)^2)'_x}{(1 + (3 + x^2)^2)^2} = \\ &= \frac{2(1 + 3^2) - 0}{(1 + 3^2)^2} = \frac{1}{5}. \end{aligned}$$

Similarly, we find for the second term in $\Phi'(x)$

$$\left(\frac{-\sin x}{1 + (2 + \cos x)^2} \right)'_x = -\frac{1}{10}.$$

Check it yourself.

4) Now, using (6.1), we write the expansion for the entire numerator

$$\Phi(x) = \Phi(0) + \Phi'(0)x + \frac{1}{2}\Phi''(0)x^2 + o(x^2).$$

Substituting, we get

$$\Phi(x) = 0 + 0x + \frac{1}{2} \left(\frac{1}{5} - \left(-\frac{1}{10} \right) \right) x^2 + o(x^2) = \frac{3}{20}x^2 + o(x^2).$$

$$\text{Finally } A = \lim_{x \rightarrow 0} \frac{\frac{3}{20}x^2 + o(x^2)}{-x^2 + o(x^2)} = -\frac{3}{20}.$$

Example 6.5 Find the first three terms of the expansion in the Maclaurin formula for the function $\operatorname{tg} x$.

Solution. 1) We use tabular expansions for the functions $\sin x$ and $\cos x$, as well as the expansion with undetermined coefficients for the *odd* function $\operatorname{tg} x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5),$$

$$\operatorname{tg} x = ax + bx^3 + cx^5 + o(x^5).$$

- 2) We find the values of the undetermined coefficients using the equality

$$\cos x \cdot \operatorname{tg} x = \sin x ,$$

which in the introduced notations has the form

$$\begin{aligned} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)\right) (ax + bx^3 + cx^5 + o(x^5)) = \\ = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right) . \end{aligned}$$

- 3) Multiplying the polynomials on the left side and equating the coefficients at equal powers of x on both sides, we obtain the values of the desired coefficients

$$a = 1, \quad b = \frac{1}{3}, \quad c = \frac{2}{15}.$$

Let us now consider examples of calculating a limit of the type specified in item 2) of Theorem 6.1.

Example 6.6 Find $A = \lim_{x \rightarrow 0} \left(\sqrt{1+2x} \cdot \cos x - \frac{\arcsin x}{1+x} \right)^{\frac{x}{(1-\operatorname{ch} x)^2}}$.

Solution. 1) Since the exponent is simpler than the base, we begin to find an expansion for the exponent. We have

$$1 - \operatorname{ch} x = -\frac{1}{2}x^2 + o(x^2) \implies (1 - \operatorname{ch} x)^2 = \frac{1}{4}x^4 + o(x^4).$$

Then

$$\frac{x}{(1 - \operatorname{ch} x)^2} = \frac{x}{\frac{1}{4}x^4 + o(x^4)} = \frac{1}{\frac{1}{4}x^3 + o(x^3)}.$$

Therefore, $m = 3$.

2) We obtain the expansion of the base as follows. We have

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^3),$$

$$\begin{aligned}\sqrt{1+2x} &= 1 + \frac{1}{2}2x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}4x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6}8x^3 + o(x^3) = \\ &= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + o(x^3)\end{aligned}$$

Then

$$\begin{aligned}\sqrt{1+2x} \cdot \cos x &= \\ &= \left(1 - \frac{1}{2}x^2 + o(x^3)\right) \cdot \left(x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + o(x^3)\right) = \\ &= 1 + x - x^2 + o(x^3).\end{aligned}$$

Similarly, taking into account table formulas 13) and 6),
from

$$\arcsin x = 1 + \frac{1}{6}x^3 + o(x^3)$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3),$$

we get

$$\begin{aligned} \frac{\arcsin x}{1+x} &= \\ &= \left(1 + \frac{1}{6}x^3 + o(x^3)\right) \cdot (1 - x + x^2 - x^3 + o(x^3)) = \\ &= x - x^2 - \frac{5}{6}x^3 + o(x^3). \end{aligned}$$

As a result, for the base we have $1 + \frac{5}{6}x^3 + o(x^3)$.

3) Finally, by item 2) of Theorem 6.1 we obtain

$$A = \lim_{x \rightarrow 0} \left(1 + \frac{5}{6}x^3 + o(x^3) \right)^{\frac{1}{\frac{1}{4}x^3 + o(x^3)}} = e^{\frac{5}{6} \cdot 4} = e^{\frac{10}{3}}.$$

Note that under the conditions of Theorem 6.1 $A = e^B$, where

$$B = \lim_{x \rightarrow 0} \frac{\ln(1 + f(x))}{g(x)} = \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \ln A. \quad (6.3)$$

Let's consider another example of uncertainty disclosure « 1^∞ ».

Example 6.7. Find $A = \lim_{x \rightarrow 0} \left(\frac{2x}{\sin 2x} - \frac{2}{3}x^2 \right)^{\frac{x^2}{x^2 - \arctg x^2}}$.

Solution. 1) Let's start with constructing the expansion for the exponent. We have

$$\frac{x^2}{x^2 - \arctg x^2} = \frac{x^2}{x^2 - \left(x^2 - \frac{x^6}{3} + o(x^9) \right)} = \frac{1}{\frac{x^4}{3} + o(x^7)}.$$

That is, $m = 4$.

- 2) To obtain the Maclaurin formula for the base, we first expand the fraction to $o(x^4)$

$$\begin{aligned} \frac{2x}{\sin 2x} &= \frac{2x}{2x - \frac{1}{6}8x^3 + \frac{1}{120}32x^5 + o(x^6)} = \\ &= \frac{1}{1 - \left(\frac{2}{3}x^2 - \frac{2}{15}x^4 + o(x^5)\right)} = \\ &= 1 + \left(\frac{2}{3}x^2 - \frac{2}{15}x^4 + o(x^5)\right) + \left(\frac{2}{3}x^2 + o(x^3)\right)^2 + o(x^5) = \end{aligned}$$

As a result, the Maclaurin representation of the base will be

$$\frac{2x}{\sin 2x} - \frac{2}{3}x^2 = 1 + \frac{14}{45}x^4 + o(x^5).$$

Finally, applying the second assertion of Theorem 6.1, we obtain $A = e^{\frac{14}{45} \cdot 3} = e^{\frac{14}{15}}.$