Determinants

Consider a set consisting of natural numbers $1, 2, 3, \ldots, n$. We will denote the *permutations* of these numbers (that is, their sequential writing in some order without gaps or repetitions) as $\{k_1, k_2, k_3, \ldots, k_n\}$. Recall that the total number of such different permutations is $n!$.

Definition	We will say that the numbers k_i and k_j form a *disorder* in a permutation (a *violation of order*, or *inversion*), if for $i < j$ holds $k_i > k_j$.

We will denote the total number of disorders in a permutation $\{k_1, k_2, k_3, \ldots, k_n\}$ by $Б(k_1, k_2, \ldots, k_n)$. For example, $Б(3, 1, 4, 2) = 3$.

Let a square matrix be given

$$\|A\| = \left\| \begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \dots & \alpha_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \dots & \alpha_{nn} \end{matrix} \right\| = \| \alpha_{ij} \|; \; i, j = [1, n].$$

Definition The *determinant* of a square matrix $\|A\|$ of size $n \times n$ is the number $\det\|A\|$ obtained by the formula

$$\det\| A \| = \sum_{\{k_1, k_2, k_3, \dots, k_n\}} (-1)^{Б(k_1, k_2, k_3, \dots, k_n)} \alpha_{1k_1} \alpha_{2k_2} \dots \alpha_{nk_n} ,$$

where $\{k_1, k_2, k_3, \dots, k_n\}$ are all possible different permutations formed from the column numbers of the matrix $\| A \|$.

Since this definition specifies that the sum is taken over *all possible different* permutations, the number of terms is $n!$

It also follows from this definition that each term contains as a factor one matrix element from each column and each row..

Properties of determinants

Theorem. **When transposing a square matrix, its determinant does not change.**

Corollary **Any property of the determinant of a matrix formulated for its columns is true for its rows, and vice versa.**

Theorem **When two columns of a matrix are transposed, the sign of its determinant changes to the opposite.**

Corollary **The determinant of a matrix containing two identical columns is equal to zero.**

Theorem (linear property of the determinant). **If the k-th column of a matrix is given as a linear combination of some "*new*" columns, then its determinant can be represented as the same linear combination of determinants of matrices whose k-th columns are the corresponding "*new*" columns from the original linear combination.**

Corollary. **When calculating the determinant, a common factor can be taken out of a matrix column.**

Corollary. **If a linear combination of the remaining columns is added to some matrix column, then the determinant will not change.**

Theorem **The determinant of a product of matrices of size $n\times n$ is equal to the product of their determinants, that is,**

$$\det(\|A\|\|B\|)=\det\|A\|\cdot\det\|B\|.$$

Decomposition of determinants

We select rows with numbers $i_1, i_2, \ldots, i_k$ and columns with numbers $j_1, j_2, \ldots, j_k$ in a square matrix $\|A\|$ of order n, where $1 \le k \le n$.

Definition	The determinant of a square matrix of order k, formed by elements standing at the intersection of rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$, is called a *minor of order* k and is denoted by $M^{j_1, j_2, \ldots, j_k}_{i_1, i_2, \ldots, i_k}$. The determinant of a square matrix of order $n-k$, formed by elements remaining after deleting rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$, is called a *minor complementary* to the minor $M^{j_1, j_2, \ldots, j_k}_{i_1, i_2, \ldots, i_k}$, and is denoted by $\overline{M}^{j_1, j_2, \ldots, j_k}_{i_1, i_2, \ldots, i_k}$.

Let's select the i-th row and j-th column of the matrix $\|A\|$, at the intersection of which the element α_{ij} is located. We then delete the selected row and column, and consider a square matrix $\|A^{+}\|$ of size $(n-1)\times(n-1)$.

Definition The determinant of a matrix $\|A^{+}\|$ is called the *complementary minor* $\overline{M}_i^j$ of the element α_{ij}.

Let's group all $(n-1)!$ the terms containing the element in the definition of the determinant of a matrix $\|A\|$ and take it out of the brackets. We get an expression of the form

$$\det\|A\| = \alpha_{ij}D_{ij} + \dots .$$

Definition The number D_{ij} is called the *algebraic complement* of the element α_{ij}.

Note that according to this definition, the following equalities hold

$$\det\|A\| = \sum_{j=1}^{n}\alpha_{ij}D_{ij} = \sum_{k=1}^{n}\alpha_{kj}D_{kj} \qquad \forall j=[1,n],\ \forall i=[1,n],$$

which can be used to calculate the determinants of square matric

Equalitiesa

$$\det\|A\| = \sum_{j=1}^{n} \alpha_{ij} D_{ij} = \sum_{k=1}^{n} \alpha_{kj} D_{kj} \qquad \forall j = [1,n],\ \forall i = [1,n],$$

can be used to calculate determinants of square matrices by finding the values of algebraic complements using the following

Theorem **The equalities are true** $D_{ij} = (-1)^{i+j} \overline{M}_i^{\,j}$ **.**

Corollary **The expansion of the determinant by the** i**-th column has the form**

$$\det\| A \| = \sum_{k=1}^{n} (-1)^{k+i} \alpha_{ki} \overline{M}_k^{\,i}$$

or

$$\det\| A \| = \sum_{k=1}^{n} (-1)^{k+i} M_k^{\,i} \overline{M}_k^{\,i} \,.$$

For practical applications, the following theorem is especially useful

Theorem. **For any square matrix $\|A\|$, the equality holds**

$$\sum_{i=1}^{n} \alpha_{ij} D_{is} = \delta_{js} \cdot \Delta ,$$

where $\Delta = \det\|A\|$ **and** $\delta_{js} = \begin{cases} 1 \ , \ j = s, \\ 0 \ , \ j \neq s \end{cases}$ **is the *Kronecker delta*.**

Corollary **If a square matrix $\|A\|$ is non-singular, then the elements of its inverse matrix $\|A\|^{-1}$ are the number**

$$\beta_{ij} = \frac{(-1)^{i+j}\,\overline{M}^{\,i}_{j}}{\Delta} \qquad i, j = [1, n] .$$

Task 11.01. *Show without direct calculation that the determinant*

$$\Delta = \det \begin{Vmatrix} 1 & 2 & 3 & 1 \\ 3 & 4 & 9 & 1 \\ 5 & 1 & 5 & 2 \\ 7 & 9 & 3 & 1 \end{Vmatrix}.$$

is divisible by 59 without remainder if it is known that the equalities are true

$$\begin{aligned} 59\times 23 &= 1357, \\ 59\times 41 &= 2419, \\ 59\times 67 &= 3953, \\ 59\times 19 &= 1121. \end{aligned}$$

Solution: 1). The value of the determinant will not change if the matrix is transposed. Therefore

$$\Delta = \det\begin{Vmatrix} 1 & 2 & 3 & 1 \\ 3 & 4 & 9 & 1 \\ 5 & 1 & 5 & 2 \\ 7 & 9 & 3 & 1 \end{Vmatrix}. = \det\begin{Vmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 1 & 9 \\ 3 & 9 & 5 & 3 \\ 1 & 1 & 2 & 1 \end{Vmatrix}.$$

2). We perform the following transformations with the resulting matrix, which do not change the value of its determinant. We successively add to the last column

the third column multiplied by 10,

then the second column multiplied by 100,

and finally the first column multiplied by 1000.

As a result, we get:

$$\Delta = \det\begin{Vmatrix} 1 & 3 & 5 & 1357 \\ 2 & 4 & 1 & 2419 \\ 3 & 9 & 5 & 3953 \\ 1 & 1 & 2 & 1121 \end{Vmatrix}.$$

3). According to the condition of the problem and the consequence of the linear property of the determinant, the common factor 59 can be taken out of the fourth column. This gives

$$\Delta = 59 \cdot \det \begin{Vmatrix} 1 & 3 & 5 & 23 \\ 2 & 4 & 1 & 41 \\ 3 & 9 & 5 & 67 \\ 1 & 1 & 2 & 19 \end{Vmatrix} = 59 \cdot K ,$$

where, by virtue of the definition of the determinant, the number K is an integer. Therefore, the number Δ will also be an integer.

Solution is found

Task 11.02. *Find the determinant of the matrix of order* n :

$$\Delta_n = \det \begin{Vmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & x \end{Vmatrix}.$$

Solution: 1) Note that the sum of the elements of each column of the matrix is the same and equal to $x + a(n-1)$. Therefore, adding the sum of the remaining rows to the first row and taking the common factor out of the first row, we get a matrix with the same determinant

$$\Delta_n = (x + a(n-1)) \cdot \det \begin{Vmatrix} 1 & 1 & 1 & \dots & 1 \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & x \end{Vmatrix} .$$

2). Subtracting from each row, starting from the second, the first row multiplied by a, we obtain

$$\Delta_n = (x + a(n-1)) \cdot \det \begin{Vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & x-a & 0 & \ldots & 0 \\ 0 & 0 & x-a & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & x-a \end{Vmatrix}.$$

3). Successively applying the formula for expanding the determinant by the first column, we arrive at the expression

$$\Delta_n = (x + a(n-1))(x-a)^{n-1} \quad .$$

Solution is found

Task 11.03. *Find the determinant of the n-th order matrix:*

$$\Delta_n = \det\begin{Vmatrix} 2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{Vmatrix}.$$

Sollution: 1) Note that for $n=3$, expanding the determinant along the first row we have

$$\Delta_3=\det\begin{Vmatrix}2&1&0\\1&2&1\\0&1&2\end{Vmatrix}=2\cdot\det\begin{Vmatrix}2&1\\1&2\end{Vmatrix}-\det\begin{Vmatrix}1&1\\0&2\end{Vmatrix}=2\cdot3-2=4\,,$$

and for $n=4$, similarly we obtain that

$$\Delta_4=\det\begin{Vmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{Vmatrix}=2\cdot\det\begin{Vmatrix}2&1&0\\1&2&1\\0&1&2\end{Vmatrix}-1\cdot\det\begin{Vmatrix}1&1&0\\0&2&1\\0&1&2\end{Vmatrix}=$$

$$=2\cdot\Delta_3-1\cdot\det\begin{Vmatrix}2&1\\1&2\end{Vmatrix}=2\cdot4-3=5\,.$$

2). Now let's write the expansion Δ_n along the first row

$$\Delta_n = 2\cdot\det\begin{Vmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{Vmatrix} - 1\cdot\det\begin{Vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{Vmatrix} =$$

$$= 2\cdot\Delta_{n-1} - 1\cdot\Delta_{n-2}$$

3) Let us consider the recurrence relation $\Delta_n = 2\cdot\Delta_{n-1} - 1\cdot\Delta_{n-2}$ as a difference equation, considering Δ_n as an unknown function of *n*.

Its solution is any linear function of the form $\Delta_n = a\,n + b,$ where *a* and *b* are some constants. Indeed,

$$an + b = 2\cdot(\ a(n-1) + b\ ) - (\ a(n-2) + b\ ) \qquad \Rightarrow \qquad 0 = 0\,.$$

We will find the values of the constants *a* and *b* from the conditions

$$\begin{cases} \Delta_3 = 4, \\ \Delta_4 = 5 \end{cases} \qquad \Rightarrow \qquad \begin{cases} a\cdot 3 + b = 4, \\ a\cdot 4 + b = 5 \end{cases} \qquad \Rightarrow \qquad a = b = 1.$$

Thus, $\Delta_n = n + 1$.

Solution is found