Second-order surfaces (quadrics) in space

Let an Cartesian coordinate system with an *orthonormal* basis be given in space $\{O,\vec{e}_1,\vec{e}_2,\vec{e}_3\}$.

Definition. A surface S is an *algebraic second-order surface* (or *quadric*) if its equation in the given coordinate system can have the form

$$\begin{aligned} &A_{11}x^2 + A_{22}y^2 + A_{33}z^2 + \\ &+ 2A_{12}xy + 2A_{13}xz + 2A_{23}yz + \\ &+ 2A_{14}x + 2A_{24}y + 2A_{34}z + A_{44} = 0, \end{aligned}$$

where the numbers $A_{11}, A_{22}, A_{33}, A_{12}, A_{13}, A_{23}$ are not equal to zero simultaneously, and x, y and z are the coordinates of the radius vector of a point belonging to S .

As in the plane case, the coefficients of the surface equation depend on the choice of coordinate system. Therefore, when studying the properties of second-order surfaces (quadrics), it is advisable to first switch to the coordinate system for which the surface equation turns out to be the *simplest*.

Theorem **For each second-order surface, there is an Cartesian coordinate system $\{O', \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ with an *orthonormal* basis in which the equation of this surface has one of the following seventeen standard forms:**

Empty sets	***Points, lines and planes***	***Cylindrical and conical surfaces***
$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=-1$	*Isolated point* $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=0$	*Elliptic cylinder* $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1\ \forall z'$
$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=-1\ \forall z'$	*Line* $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=0\ \forall z'$	*Hyperbolic cylinder* $\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=1\ \forall z'$
$x'^2=-a^2\ \ \forall\, y', z'$	*Pair of intersecting planes* $\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=0\ \forall z'$	*Parabolic cylinder* $y'^2=2px'\ \forall z'$
	Pair of parallel or coinciding planes $x'^2=a^2\ \forall\, y', z'$ $x'^2=0\ \ \forall\, y', z'$	*Conical surface* $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-\frac{z'^2}{c^2}=0$

Nonsingular surfaces		
Ellipsoids	*Paraboloids*	*Hyperboloids*
Ellipsoid	*Elliptic paraboloid*	*One-sheet hyperboloid*
$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1$	$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=2z'$	$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-\frac{z'^2}{c^2}=1$
	Hyperbolic paraboloid	*Two-sheet hyperboloid*
	$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=2z'$	$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-\frac{z'^2}{c^2}=1$

and $a>0,\ b>0,\ c>0,\ p>0$.

Rectilinear generators of 2nd order surfaces (quadrics)

Cones, cylinders, planes and lines obviously have rectilinear generators. In addition, hyperbolic paraboloids and single-sheet hyperboloids also have them.

1) we write the equation of a hyperbolic paraboloid in the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z\ ,$$

then we can conclude that for any values of the parameter α , points lying on the lines

$$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2\alpha, \\ \alpha\left(\dfrac{x}{a}-\dfrac{y}{b}\right)=z \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{x}{a}-\dfrac{y}{b}=2\alpha, \\ \alpha\left(\dfrac{x}{a}+\dfrac{y}{b}\right)=z, \end{cases}$$

also belong to the hyperbolic paraboloid, since the term-by-term multiplication of the equations of the planes defining these lines yields the equation of the hyperbolic paraboloid.

Note that for each point of the hyperbolic paraboloid, there is a pair of lines passing through this point and lying entirely on the hyperbolic paraboloid. The equations of these lines can be obtained (up to some common non-zero factor) by selecting specific values of the parameter α .

2). A one-sheet hyperboloid has two families of rectilinear generators. Having written the equation of this surface in the form

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=1-\frac{y^2}{b^2},$$

we can come to the conclusion that for any non-zero simultaneously α and β, the points lying on the lines

$$\begin{cases}\alpha\left(\frac{x}{a}+\frac{z}{c}\right)=\beta\left(1-\frac{y}{b}\right),\\ \beta\left(\frac{x}{a}-\frac{z}{c}\right)=\alpha\left(1+\frac{y}{b}\right)\end{cases} \quad \text{and} \quad \begin{cases}\alpha\left(\frac{x}{a}+\frac{z}{c}\right)=\beta\left(1+\frac{y}{b}\right),\\ \beta\left(\frac{x}{a}-\frac{z}{c}\right)=\alpha\left(1-\frac{y}{b}\right)\end{cases}$$

will also belong to the one-sheet hyperboloid, since the term-by-term multiplication of the equations of the planes defining these lines yields the equation of the one-sheet hyperboloid.

That is, for each point of the one-sheet hyperboloid there is a pair of lines passing through this point and lying entirely on the one-sheet hyperboloid. The equations of these lines can be obtained by selecting specific values of α and β.

Task 9.1. *Find the rectilinear generators of the surface* $\frac{x^2}{9}-y^2=2z$ *passing through the point* $(-3,1,0)$.

Solution: 1) Represent the equation of a hyperbolic paraboloid in the form $\left(\frac{x}{3}-y\right)\left(\frac{x}{3}+y\right)=2z$, which is a consequence of each of the following two systems defining straight lines

$$\begin{cases}\frac{x}{3}+y=2\alpha,\\ \alpha\left(\frac{x}{3}-y\right)=z\end{cases}\quad\text{and}\quad\begin{cases}\frac{x}{3}-y=2\alpha,\\ \alpha\left(\frac{x}{3}+y\right)=z.\end{cases}$$

2) For the first family of straight lines, the condition of passing through the point $(-3,1,0)$ has the form $\alpha=0$. Then the desired straight line will be

$$\begin{cases}x+3y=0,\\ z=0\end{cases}\quad\text{or}\quad\begin{cases}x=-3\tau,\\ y=\tau,\\ z=0.\end{cases}$$

3) Для второго семейства прямых условие прохождения через точку $(-3,1,0)$ записывается аналогично: $\alpha=-1$. Искомая прямая в этом случае есть For the second family of straight lines, the condition of passing through the point $(-3,1,0)$ is written similarly: $\alpha=-1$. The desired straight line in this case is

$$\begin{cases}\frac{x}{3}+y+z=0,\\ \frac{x}{3}-y=-2\end{cases}\quad\text{or}\quad\begin{cases}x=-6+3\tau,\\ y=\tau,\\ z=2-2\tau.\end{cases}$$

Solution is found

Construction of quadrics

The deduction of equations of surfaces can be performed by using their geometric properties. For example, a conical surface is defined using a homothety relative to its vertex.

We will illustrate this for the case of a cylindrical surface.

Task 9.2. *Find the equation of a cylinder circumscribed around two following spheres* .$x^2+y^2+z^2=36$ *and* $(x-1)^2+(y+1)^2+(z-2)^2=36$.

Solution: 1) Obviously, the axis of the desired cylinder is a straight line passing through $\vec r_0$ - the origin, with a direction vector $\|\vec a\| = \left\|\begin{matrix}1\\-1\\2\end{matrix}\right\|$ (a vector between the centers of the spheres)

2) Let an arbitrary point $\vec\rho$ on the surface of the desired cylinder with $\|\vec\rho\| = \left\|\begin{matrix}x\\y\\z\end{matrix}\right\|$.

Since the distance from any point of the cylindrical surface to its axis is a constant R, then (according to the well-known formula for the distance from a point $\vec\rho$ to a straight line $\vec r = \vec r_0 + \tau\,\vec a$)

$$R = \frac{\left|[\vec\rho - \vec r_0, \vec a]\right|}{|\vec a|} \qquad \Rightarrow \qquad R^2|\vec a|^2 = \left|[\vec\rho - \vec r_0, \vec a]\right|^2.$$

3) In our case we have: $R=6$ and $|\,\vec{a}\,|^2 = 6$, and

$$\left|[\vec{\rho}-\vec{r}_0,\vec{a}]\right| = \left|\det\begin{Vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ x & y & z\\ 1 & -1 & 2\end{Vmatrix}\right| \Rightarrow$$

$$\Rightarrow \quad \left|[\vec{\rho}-\vec{r}_0,\vec{a}]\right|^2 = (2y+z)^2+(2x-z)^2+(x+y)^2.$$

Where we get the desired equation of the surface

$$(2y+z)^2+(2x-z)^2+(x+y)^2 = 216.$$

4) Outwardly, this equation does not look much like the canonical equation of a cylinder, rather it is the equation of an ellipsoid, since when replacing $\begin{cases} x' = 2y+z,\\ y' = 2x-z,\\ z' = x+y\end{cases}$ we get $x'^2+y'^2+z'^2 = 216.$

However, we note that these formulas are not coordinate replacement formulas (i.e. transition formulas) since for the matrix of such a replacement $\det\begin{Vmatrix}0 & 2 & 1\\ 2 & 0 & -1\\ 1 & 1 & 0\end{Vmatrix} = 0$ and it cannot be a transition matrix!

In our case, the left side of the resulting equation, due to equality $2z' = x'+y'$, can be represented as

$$3x'^2+2x'y'+3y'^2 = 432 \qquad \forall z' ,$$

which (show this yourself) is the equation of a straight, circular cylindrical surface.

Solution is found

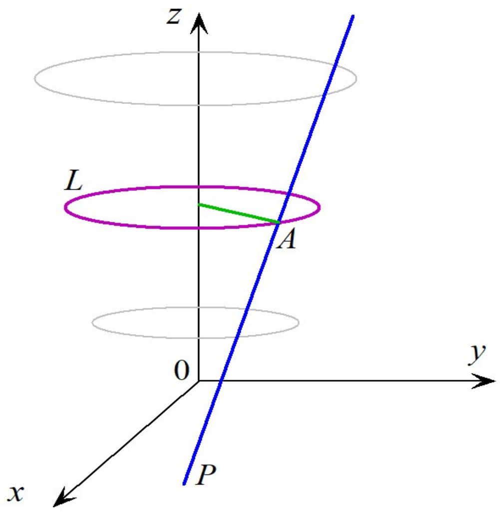
Surfaces of revolution

Definition A set of points whose coordinates satisfy the equation

$$F(\pm\sqrt{x^2+y^2}, z) = 0,$$

is called a surface of revolution around an axis Oz.

A rotation of lines of the second order (conic) around some axis is not always a surface of the second order (quadric). For example, a rotation of a line $(x-1)^2 + y^2 = 1$ around an axis Ox gives an *ellipsoid*, and a rotation around an axis Oy gives a *torus*.



Task 9.3. Find the equation of the surface obtained by rotating a line $\begin{cases} x = 2\tau, \\ y = 4, \\ z = \tau \end{cases}$ around an axis Oz and determine its type.

Solution: 1) The given line P and the axis of rotation Oz form a pair of intersecting lines. We choose A − some point on the sought surface of rotation with coordinates $\left\| \begin{matrix} x \\ y \\ z \end{matrix} \right\|$.

This point belongs to the circle L, therefore, according to the Pythagorean theorem

$$x^2 + y^2 = R^2(z),$$

where $R(z)$ is the radius of the circle L.

2) On the other hand, the point A belongs to the given line P. This means that for its co-ordinates $\begin{cases} x = 2z \\ y = 4 \end{cases}$. From which we obtain that $R^2(z) = 16 + 4z^2$.

Consequently, the desired equation will have the form $x^2 + y^2 = 16 + 4z^2$ or

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} - \frac{z^2}{2^2} = 1.$$

This means that the desired surface of revolution is a *one-sheet hyperboloid*.

Solution is found