

Second-order surfaces (quadrics) in space

Let an Cartesian coordinate system with an *orthonormal* basis be given in space $\{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Definition A surface S is an *algebraic second-order surface* (or *quadric*) if its equation in the given coordinate system can have the form

$$\begin{aligned} &A_{11}x^2 + A_{22}y^2 + A_{33}z^2 + \\ &+ 2A_{12}xy + 2A_{13}xz + 2A_{23}yz + \\ &+ 2A_{14}x + 2A_{24}y + 2A_{34}z + A_{44} = 0, \end{aligned}$$

where the numbers $A_{11}, A_{22}, A_{33}, A_{12}, A_{13}, A_{23}$ are not equal to zero simultaneously, and x , y and z are the coordinates of the radius vector of a point belonging to S .

As in the plane case, the coefficients of the surface equation depend on the choice of coordinate system. Therefore, when studying the properties of second-order surfaces (quadrics), it is advisable to first switch to the coordinate system for which the surface equation turns out to be the *simplest*.

Theorem For each second-order surface, there is an Cartesian coordinate system $\{O', \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ with an *orthonormal* basis in which the equation of this surface has one of the following seventeen standard forms:

<i>Empty sets</i>	<i>Points, lines and planes</i>	<i>Cylindrical and conical surfaces</i>
$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = -1$	<p><i>Isolated point</i></p> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 0$	<p><i>Elliptic cylinder</i></p> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad \forall z'$
$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = -1 \quad \forall z'$	<p><i>Line</i></p> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 0 \quad \forall z'$	<p><i>Hyperbolic cylinder</i></p> $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1 \quad \forall z'$
$x'^2 = -a^2 \quad \forall y', z'$	<p><i>Pair of intersecting planes</i></p> $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 0 \quad \forall z'$	<p><i>Parabolic cylinder</i></p> $y'^2 = 2px' \quad \forall z'$
	<p><i>Pair of parallel or coinciding planes</i></p> $x'^2 = a^2 \quad \forall y', z'$ $x'^2 = 0 \quad \forall y', z'$	<p><i>Conical surface</i></p> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 0$

<i>Nonsingular surfaces</i>		
<i>Ellipsoids</i>	<i>Paraboloids</i>	<i>Hyperboloids</i>
<i>Ellipsoid</i> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1$	<i>Elliptic paraboloid</i> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 2z'$ <i>Hyperbolic paraboloid</i> $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 2z'$	<i>One-sheet hyperboloid</i> $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 1$ <i>Two-sheet hyperboloid</i> $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 1$

and $a > 0, b > 0, c > 0, p > 0$.

Rectilinear generators of 2nd order surfaces (quadrics)

Cones, cylinders, planes and lines obviously have rectilinear generators. In addition, hyperbolic paraboloids and single-sheet hyperboloids also have them.

1) we write the equation of a hyperbolic paraboloid in the form

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 2z,$$

then we can conclude that for any values of the parameter α , points lying on the lines

$$\left\{ \begin{array}{l} \frac{x}{a} + \frac{y}{b} = 2\alpha, \\ \alpha\left(\frac{x}{a} - \frac{y}{b}\right) = z \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \frac{x}{a} - \frac{y}{b} = 2\alpha, \\ \alpha\left(\frac{x}{a} + \frac{y}{b}\right) = z, \end{array} \right.$$

also belong to the hyperbolic paraboloid, since the term-by-term multiplication of the equations of the planes defining these lines yields the equation of the hyperbolic paraboloid.

Note that for each point of the hyperbolic paraboloid, there is a pair of lines passing through this point and lying entirely on the hyperbolic paraboloid. The equations of these lines can be obtained (up to some common non-zero factor) by selecting specific values of the parameter α .

2). A one-sheet hyperboloid has two families of rectilinear generators. Having written the equation of this surface in the form

$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = 1 - \frac{y^2}{b^2},$$

we can come to the conclusion that for any non-zero simultaneously α and β , the points lying on the lines

$$\begin{cases} \alpha\left(\frac{x}{a} + \frac{z}{c}\right) = \beta\left(1 - \frac{y}{b}\right), \\ \beta\left(\frac{x}{a} - \frac{z}{c}\right) = \alpha\left(1 + \frac{y}{b}\right) \end{cases} \quad \text{and} \quad \begin{cases} \alpha\left(\frac{x}{a} + \frac{z}{c}\right) = \beta\left(1 + \frac{y}{b}\right), \\ \beta\left(\frac{x}{a} - \frac{z}{c}\right) = \alpha\left(1 - \frac{y}{b}\right) \end{cases}$$

will also belong to the one-sheet hyperboloid, since the term-by-term multiplication of the equations of the planes defining these lines yields the equation of the one-sheet hyperboloid.

That is, for each point of the one-sheet hyperboloid there is a pair of lines passing through this point and lying entirely on the one-sheet hyperboloid. The equations of these lines can be obtained by selecting specific values of α and β .

Task 9.1. Find the rectilinear generators of the surface $\frac{x^2}{9} - y^2 = 2z$ passing through the point $(-3, 1, 0)$.

Solution: 1) Represent the equation of a hyperbolic paraboloid in the form $\left(\frac{x}{3} - y\right)\left(\frac{x}{3} + y\right) = 2z$, which is a consequence of each of the following two systems defining straight lines

$$\begin{cases} \frac{x}{3} + y = 2\alpha, \\ \alpha\left(\frac{x}{3} - y\right) = z \end{cases} \quad \text{and} \quad \begin{cases} \frac{x}{3} - y = 2\alpha, \\ \alpha\left(\frac{x}{3} + y\right) = z. \end{cases}$$

2) For the first family of straight lines, the condition of passing through the point $(-3, 1, 0)$ has the form $\alpha = 0$. Then the desired straight line will be

$$\begin{cases} x + 3y = 0, \\ z = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = -3\tau, \\ y = \tau, \\ z = 0. \end{cases}$$

3) Для второго семейства прямых условие прохождения через точку $(-3, 1, 0)$ записывается аналогично: $\alpha = -1$. Искомая прямая в этом случае есть For the second family of straight lines, the condition of passing through the point $(-3, 1, 0)$ is written similarly: $\alpha = -1$. The desired straight line in this case is

$$\begin{cases} \frac{x}{3} + y + z = 0, \\ \frac{x}{3} - y = -2 \end{cases} \quad \text{or} \quad \begin{cases} x = -6 + 3\tau, \\ y = \tau, \\ z = 2 - 2\tau. \end{cases}$$

Solution is found

Construction of quadrics

The deduction of equations of surfaces can be performed by using their geometric properties. For example, a conical surface is defined using a homothety relative to its vertex.

We will illustrate this for the case of a cylindrical surface.

Task 9.2. Find the equation of a cylinder circumscribed around two following spheres
 $x^2 + y^2 + z^2 = 36$ and $(x-1)^2 + (y+1)^2 + (z-2)^2 = 36$.

Solution: 1) Obviously, the axis of the desired cylinder is a straight line passing through \vec{r}_0 - the

origin, with a direction vector $\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ (a vector between the centers of the spheres)

2) Let an arbitrary point $\vec{\rho}$ on the surface of the desired cylinder with $\vec{\rho} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Since the distance from any point of the cylindrical surface to its axis is a constant R , then (according to the well-known formula for the distance from a point $\vec{\rho}$ to a straight line $\vec{r} = \vec{r}_0 + \tau \vec{a}$)

$$R = \frac{|[\vec{\rho} - \vec{r}_0, \vec{a}]|}{|\vec{a}|} \Rightarrow R^2 |\vec{a}|^2 = |[\vec{\rho} - \vec{r}_0, \vec{a}]|^2.$$

3) In our case we have: $R = 6$ and $|\vec{a}|^2 = 6$, and

$$|[\vec{\rho} - \vec{r}_0, \vec{a}]| = \left| \det \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x & y & z \\ 1 & -1 & 2 \end{vmatrix} \right| \Rightarrow$$

$$\Rightarrow |[\vec{\rho} - \vec{r}_0, \vec{a}]|^2 = (2y + z)^2 + (2x - z)^2 + (x + y)^2.$$

Where we get the desired equation of the surface

$$(2y + z)^2 + (2x - z)^2 + (x + y)^2 = 216.$$

4) Outwardly, this equation does not look much like the canonical equation of a cylinder, rather it is the equation of an ellipsoid, since when replacing $\begin{cases} x' = 2y + z, \\ y' = 2x - z, \\ z' = x + y \end{cases}$ we get $x'^2 + y'^2 + z'^2 = 216$.

However, we note that these formulas are not coordinate replacement formulas (i.e. transition formulas) since for the matrix of such a replacement $\det \begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 0$ and it cannot be a transition matrix!

In our case, the left side of the resulting equation, due to equality $2z' = x' + y'$, can be represented as

$$3x'^2 + 2x'y' + 3y'^2 = 432 \quad \forall z',$$

which (show this yourself) is the equation of a straight, circular cylindrical surface.

Solution is found

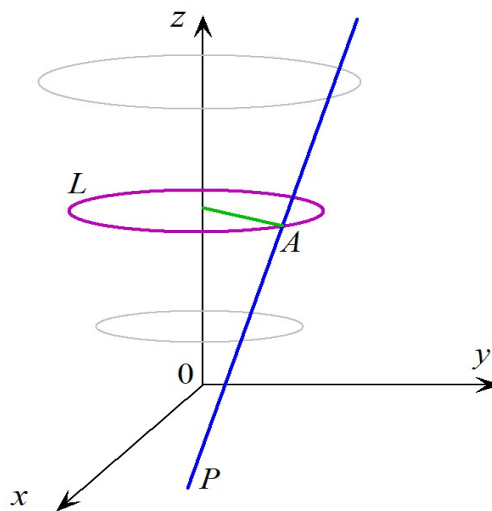
Surfaces of revolution

Definition A set of points whose coordinates satisfy the equation

$$F(\pm\sqrt{x^2 + y^2}, z) = 0,$$

is called a surface of revolution around an axis Oz .

A rotation of lines of the second order (conic) around some axis is not always a surface of the second order (quadric). For example, a rotation of a line $(x-1)^2 + y^2 = 1$ around an axis Ox gives an *ellipsoid*, and a rotation around an axis Oy gives a *torus*.



Task 9.3. Find the equation of the surface obtained by rotating a line $\begin{cases} x = 2\tau, \\ y = 4, \\ z = \tau \end{cases}$ around an axis Oz and determine its type.

Solution: 1) The given line P and the axis of rotation Oz form a pair of intersecting lines. We

choose A – some point on the sought surface of rotation with coordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

This point belongs to the circle L , therefore, according to the Pythagorean theorem

$$x^2 + y^2 = R^2(z),$$

where $R(z)$ is the radius of the circle L .

2) On the other hand, the point A belongs to the given line P . This means that for its co-ordinates $\begin{cases} x = 2z \\ y = 4 \end{cases}$. From which we obtain that $R^2(z) = 16 + 4z^2$.

Consequently, the desired equation will have the form $x^2 + y^2 = 16 + 4z^2$ or

$$\frac{x^2}{4^2} + \frac{y^2}{4^2} - \frac{z^2}{2^2} = 1.$$

This means that the desired surface of revolution is a *one-sheet hyperboloid*.

Solution is found