Second-order curves (conics) in a plane

Let an Cartesian coordinate system be given in a plane $\{O, \vec{e}_1, \vec{e}_2\}$ with *orthonormal basis*.

Definition A line L is called an *algebraic second-order line* (or *conics*) if its equation in a given coordinate system has the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

where the numbers A, B and C are not equal to zero simultaneously, and x and y are the coordinates of the radius vector of a point belonging to L.

Since the coefficients of this equation depend on the choice of coordinate system, when studying the properties of second-order lines (conics), it is advisable to try to find another orthonormal coordinate system $\{O', \vec{e}_1{}', \vec{e}_2{}'\}$ in which the equation of the line is *simpler*.

The solution to this problem is given by

Theorem **For any second-order line** (conics)**, there exists an Cartesian coordinate system (with *orthonormal basis*) in which the equation of this line (for** $a \geq b > 0,\ p > 0$ **) has one of the following nine (called *canonical*) forms:**

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Line type / ***Line class***	*Elliptic* $\Delta > 0$	*Hyperbolic* $\Delta < 0$	*Parabolic* $\Delta = 0$
Empty sets	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = -1$		$y'^2 = -a^2 \quad \forall x'$
Isolated points	$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 0$		
Coincident lines			$y'^2 = 0 \quad \forall x'$
Non-coincident lines		$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 0$	$y'^2 = a^2 \quad \forall x'$
Curves	*Ellipse* $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$	*Hyperbola* $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$	*Parabola* $y'^2 = 2px'$

where $\Delta = \det\begin{Vmatrix} A & B \\ B & C \end{Vmatrix} = AC - B^2.$

The equations of the *tangents* to an ellipse, hyperbola and parabola passing through a point (x_0, y_0) belonging to a line in the canonical coordinate system have, respectively, of the form

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \qquad \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1 \qquad \text{and} \qquad yy_0 = p(x + x_0).$$

The following problems illustrate methods for studying the properties of second-order lines (conics).

Task 8.1. *Find all points on an ellipse* $\frac{x^2}{32} + \frac{y^2}{8} = 1$ *for which the tangent is parallel to the line* $x - 2y + 4 = 0$.

Solution: 1) The equation of the tangent to a given ellipse passing through a point on the ellipse with coordinates and , is of the form: $\frac{x_0 x}{32} + \frac{y_0 y}{8} = 1$.

2) The condition of parallelism or coincidence on the plane of two lines

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2 = 0$$

has the form: $\exists \lambda \neq 0$ such that $A_1 = \lambda A_2$ and $B_1 = \lambda B_2$. In this problem, this condition gives the relations

$$\frac{x_0}{32} = \lambda \cdot 1 \quad \text{and} \quad \frac{y_0}{8} = \lambda \cdot (-2) \quad \Rightarrow$$

$$\Rightarrow \quad x_0 = 32\lambda \quad \text{and} \quad y_0 = -16\lambda .$$

3) The point with coordinates x_0 and y_0 belongs to the ellipse, therefore

$$\frac{x_0^2}{32} + \frac{y_0^2}{8} = 1 \quad \Rightarrow \quad \frac{(32\lambda)^2}{32} + \frac{(-16\lambda)^2}{8} = 1 \quad \Rightarrow \quad 64\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm\frac{1}{8} .$$

From which we obtain two points: (4, -2) and (-4, 2), which are the solution to the problem.

Solution is found

Task 8.2. *On a hyperbola* $\frac{x^2}{4} - y^2 = 1$*, find all points whose distance to one of the asymptotes is equal to three times the distance to the other.*

Solution: 1) The equation of the tangent to a given hyperbola passing through a point with coordinates x_0 and y_0 has the form: $\frac{x_0 x}{4} - y_0 y = 1$. In this case, the equations of the asymptotes of this hyperbola will be $y = \pm\frac{x}{2}$.

2) Recall that the distance from a point with coordinates x_0 and y_0 to a line of the form $Ax + By + C = 0$, in an orthonormal coordinate system is given by the formula

$$L = \frac{|\,Ax_0 + By_0 + C\,|}{\sqrt{A^2 + B^2}} \quad .$$

Therefore, the distances to the first and second asymptotes in our problem will be

$$L_1 = \frac{\left| -\dfrac{x_0}{2} + y_0 \right|}{\sqrt{\left(\dfrac{1}{2}\right)^2 + 1^2}} = \frac{|-x_0 + 2y_0|}{\sqrt{5}} \qquad \text{and} \qquad L_1 = \frac{\left| \dfrac{x_0}{2} + y_0 \right|}{\sqrt{\left(\dfrac{1}{2}\right)^2 + 1^2}} = \frac{|x_0 + 2y_0|}{\sqrt{5}} .$$

Since the condition does not specify which of the asymptotes the point is closer to, we will have to consider two cases: $L_1 = 3L_2$ and $3L_1 = L_2$.

3) Let's consider the first case. Since the point with coordinates x_0 and y_0 belongs to the hyperbola, it is necessary to solve the system of equations

$$\begin{cases} |-x_0 + 2y_0| = 3|x_0 + 2y_0|, \\ \dfrac{x_0^2}{4} - y_0^2 = 1. \end{cases}$$

The expansion of the absolute value symbols leads to two variants:

$$\begin{cases} -x_0+2y_0 = 3(x_0+2y_0), \\ \dfrac{x_0^2}{4} - y_0^2 = 1 \end{cases} \qquad \text{or} \qquad \begin{cases} x_0-2y_0 = 3(x_0+2y_0), \\ \dfrac{x_0^2}{4} - y_0^2 = 1 \end{cases}$$

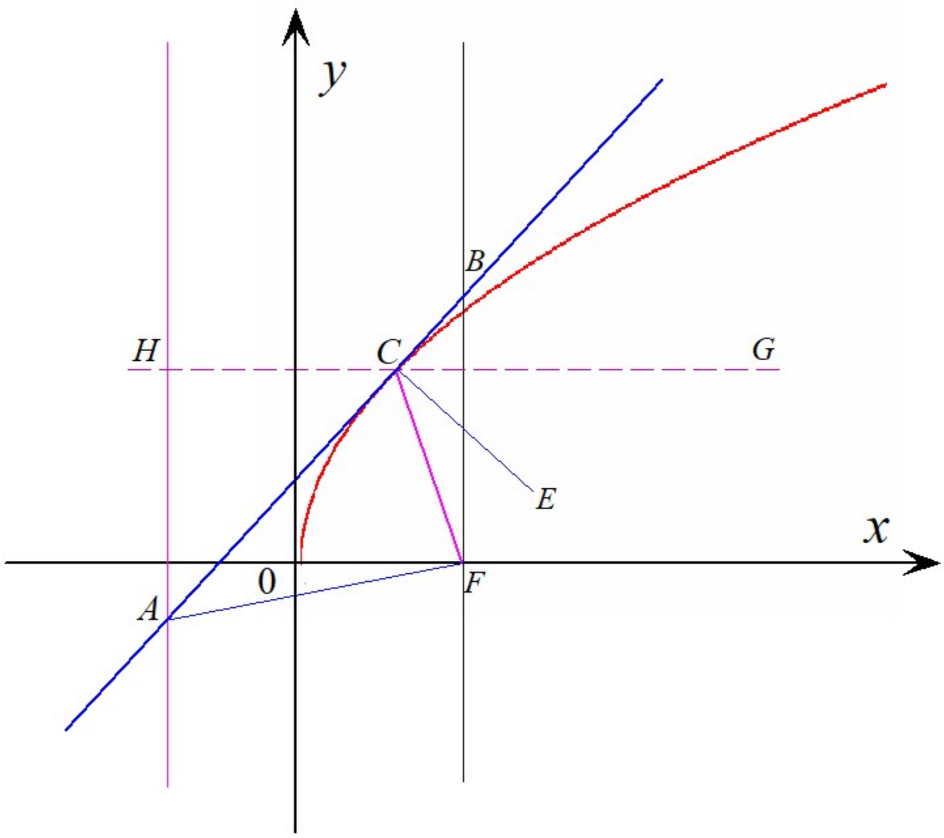
In the first variant, from the first equation we have $-x_0 = y_0$, which, by virtue of the second equation, yields $-\dfrac{3y_0^2}{4} = 1$, that is, there are no solutions here.

In the second variant, from the first equation we obtain $x_0 = -4y_0$. Then from the second equation we have $3y_0^2 = 1$, or $y_0 = \pm\dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad x_0 = \pm\dfrac{4}{\sqrt{3}}$. This yields two solutions to the problem $\left(-\dfrac{4}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ and $\left(\dfrac{4}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$.

4) The reasoning in the case $3L_1 = L_2$ is similar. They yield two solutions to the problem $\left(\dfrac{4}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ and $\left(-\dfrac{4}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$, which can also be obtained from the first pair of solutions and the symmetry of the hyperbola branches relative to the canonical coordinate axes.

Solution is found

Task 8.3. *Find out whether the statement is true:* ***the tangent to the parabola intersects the directrix and the focal chord perpendicular to the parabola axis at points equidistant from the focus****. Justify your answer.*



Explanations for the figure:

the red line is the parabola,
the blue line is the tangent,
line *AH* is the directrix,
line *FB* is the focal chord,
C is the point of tangency of the parabola,
A is the point of intersection of the tangent and the directrix,
B is the point of intersection of the tangent and the focal chord,
the line *HG* is parallel to the axis *Ox*,
the line *CE* is perpendicular to the tangent.

Let us give two versions of proof of the validity of this statement: standard geometric (using a drawing) and analytical, not using visual images.

Geometric proof

Angles *GCE* and *ECF* are equal by the optical property of a parabola and are equal to α.

Angles *ECF* and *CBF* are equal α as angles with mutually perpendicular sides

Angles *DAC* and *CBF* are equal α as intersecting at parallel lines.

Angles *ACF* and *BCG* are equal to $\beta = \frac{\pi}{2} - \alpha$.

Angle *HCA* is also equal to $\beta = \frac{\pi}{2} - \alpha$.

Then, by virtue of $HC = CF$ (since the eccentricity for a parabola is 1), triangles *AHC* and *ACF* are equal, which means angles DAC and CAF are equal .

But then angles *CAF* and *CBF* are equal α, that is, triangle ABF is isosceles and $AF = BF$.

Analytical proof

Let the point of tangency have coordinates x_0 and y_0. Then we have that $y_0^2 = 2px_0$, and the equation of the tangent will be $y_0 y = p(x + x_0)$. In addition, we take into account that the equation of the directrix is $x = -\frac{p}{2}$, and the equation of the focal chord is $x = \frac{p}{2}$. Also note that FB is equal to the modulus of the ordinate of point *B*, and *FA* is the distance between points *F* and *A*.

The focal point *F* has coordinates $\left(\frac{p}{2}, 0\right)$. Then the coordinates of point *B* are defined as follows:

$$\begin{cases} x_B = \dfrac{p}{2} \\ y_0 y_B = p(x_B + x_0) \end{cases} \Rightarrow \quad y_B = \frac{p}{y_0}\left(\frac{p}{2} + x_0\right) \quad \Rightarrow \quad y_B = \frac{p^2 + px_0}{2y_0}.$$

On the other hand, the coordinates of point A are found from the system

$$\begin{cases} x_A = -\dfrac{p}{2} \\ y_0 y_A = p(x_A + x_0) \end{cases} \Rightarrow \quad y_A = \frac{p}{y_0}\left(-\frac{p}{2} + x_0\right) \quad \Rightarrow \quad y_A = \frac{-p^2 + 2px_0}{2y_0}.$$

Since the coordinate system is orthonormal, then taking into account $y_0^2 = 2px_0$ we obtain

$$FA = \sqrt{\left(x_A - \frac{p}{2}\right)^2 + y_A^2} = \sqrt{\left(-\frac{p}{2} - \frac{p}{2}\right)^2 + \left(\frac{-p^2 + 2px_0}{2y_0}\right)^2} =$$

$$= \sqrt{\frac{4p^2y_0^2 + p^4 - 4p^3x_0 + 4p^2x_0^2}{4y_0^2}} = \sqrt{\frac{8p^3x_0 + p^4 - 4p^3x_0 + p^2x_0^2}{4y_0^2}} = \left|\frac{p^2 + px_0}{2y_0}\right| = FB.$$

Practical classification of second-order (conics) curves

I. The analysis is simplified if the equation of the second-order line has a special form.

Task 8.4. *Find out the type of the second-order line* (*without reducing its equation to canonical form*) $(x-4y+3)^2+(2x+3y-1)=0.$

Solution. Let us denote $\begin{cases} x'=x-4y+3, \\ y'=2x+3y-1, \end{cases}$. then we obtain $x'^2+y'=0$. In this case $\det\begin{Vmatrix} 1 & -4 \\ 2 & 3 \end{Vmatrix}=11\neq 0$. It will mean that these relations can be *transition formulas*. Since the *affine classification* of lines is preserved when replacing the coordinate system, the line in the condition is a *parabola*.

Solution is found

General case of a second-order line (conics)

Let $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \quad A^2 + B^2 + C^2 \neq 0$.

1) First, we note that without loss of generality we can assume that the conditions $B \geq 0$ and $A \geq C$ are satisfied. Indeed, if $B < 0$, then we can change the signs of all the coefficients in the equation of the line.

If $A < C$, then, by moving to a new orthonormal coordinate system for which $\vec{e}_1' = \vec{e}_2$; $\vec{e}_2' = \vec{e}_1$; $\vec{OO'} = \vec{o}$, we obtain the desired relation, since with such a transition, equalities $x = y'; y = x'$ hold according to the rules for constructing transition formulas. We also note that with this replacement, Δ does not change, since

$$\det\begin{Vmatrix} C & B \\ B & A \end{Vmatrix} = \det\begin{Vmatrix} A & B \\ B & C \end{Vmatrix} = \Delta .$$

2) The rotation around the origin of coordinates $\begin{cases} x = x'\cos\alpha - y'\sin\alpha, \\ y = x'\sin\alpha + y'\cos\alpha, \end{cases}$, which leads to the fact that the term $2Bxy$ disappears ($B' = 0$), is satisfied under the condition:

$$\operatorname{tg} 2\alpha = \frac{2B}{A-C} \; ; \; \alpha = \frac{1}{2}\operatorname{arctg}\frac{2B}{A-C} \quad \text{or, equivalent to it,} \quad \operatorname{tg}^2\alpha + \frac{A-C}{B}\operatorname{tg}\alpha - 1 = 0,\ B \neq 0,$$

This gives $A' = \dfrac{A+C}{2} + \dfrac{1}{2}\sqrt{4B^2 + (A-C)^2}$ and $C' = \dfrac{A+C}{2} - \dfrac{1}{2}\sqrt{4B^2 + (A-C)^2}$.

Task 8.5: *In a rectangular coordinate system, construct a line of the second order*

$$3x^2+12xy+12y^2-2x+46y+67=0\ .$$

Solution:

1°. Since $B\ge 0$, but $A\le C$, then we make a change of variables: $\begin{cases} x=y', \\ y=x'. \end{cases}$ (A)

We get the equation $12x'^2+12x'y'+3y'^2+46x'-2y'+67=0$.

2°. We rotate the coordinate system by an angle α counterclockwise. Moreover, $0\le\alpha\le\dfrac{\pi}{4}$. The formulas for this change $\begin{cases} x'=x''\cos\alpha-y''\sin\alpha, \\ y'=x''\sin\alpha+y''\cos\alpha. \end{cases}$

We find the value α from the condition: $\operatorname{tg} 2\alpha = \dfrac{2B}{A-C} = \dfrac{2\cdot 6}{12-3} = \dfrac{4}{3}$.

Using the formulas: $\dfrac{1}{\cos^2 2\alpha} = \operatorname{tg}^2 2\alpha + 1$ and $\begin{matrix} 1+\cos 2\phi = 2\cos^2\phi \\ 1-\cos 2\phi = 2\sin^2\phi \end{matrix}$,

we find that $\cos 2\alpha = \dfrac{3}{5}$ and $\begin{matrix} \cos\alpha = \dfrac{2}{\sqrt{5}} \\ \sin\alpha = \dfrac{1}{\sqrt{5}} \end{matrix}$. That is, $\begin{cases} x' = \dfrac{2}{\sqrt{5}} x'' - \dfrac{1}{\sqrt{5}} y'', \\ y' = \dfrac{1}{\sqrt{5}} x'' + \dfrac{2}{\sqrt{5}} y''. \end{cases}$ **(B)**

Since

$$A'' = \frac{A'+C'}{2} + \frac{1}{2}\sqrt{4B'^2 + (A'-C')^2} = \frac{12+3}{2} + \frac{1}{2}\sqrt{4\cdot 6^2 + (12-3)^2} = \frac{15}{2} + \frac{\sqrt{144+81}}{2} = 15$$

and $C'' = \dfrac{A'+C'}{2} - \dfrac{1}{2}\sqrt{4B'^2 + (A'-C')^2} = 0$,

then we arrive at an equation of the form:

$$15x''^2 + \frac{90}{\sqrt{5}}x'' - \frac{50}{\sqrt{5}}y'' + 67 = 0 \quad \text{or} \quad x''^2 + 2\frac{3}{\sqrt{5}}x'' - 2\frac{5}{3\sqrt{5}}y'' + \frac{67}{15} = 0\,.$$

3°. We select the full square: $x''^2 + 2\frac{3}{\sqrt{5}}x'' + \frac{9}{5} - 2\frac{5}{3\sqrt{5}}y'' + \frac{67}{15} - \frac{9}{5} = 0$ or

$$\left(x'' + \frac{3}{\sqrt{5}}\right)^2 - 2\frac{5}{3\sqrt{5}}y'' + \frac{8}{3} = 0 .$$

From which we obtain: $\left(x'' + \frac{3}{\sqrt{5}}\right)^2 = 2\frac{\sqrt{5}}{3}\left(y'' - \frac{4}{\sqrt{5}}\right)$.

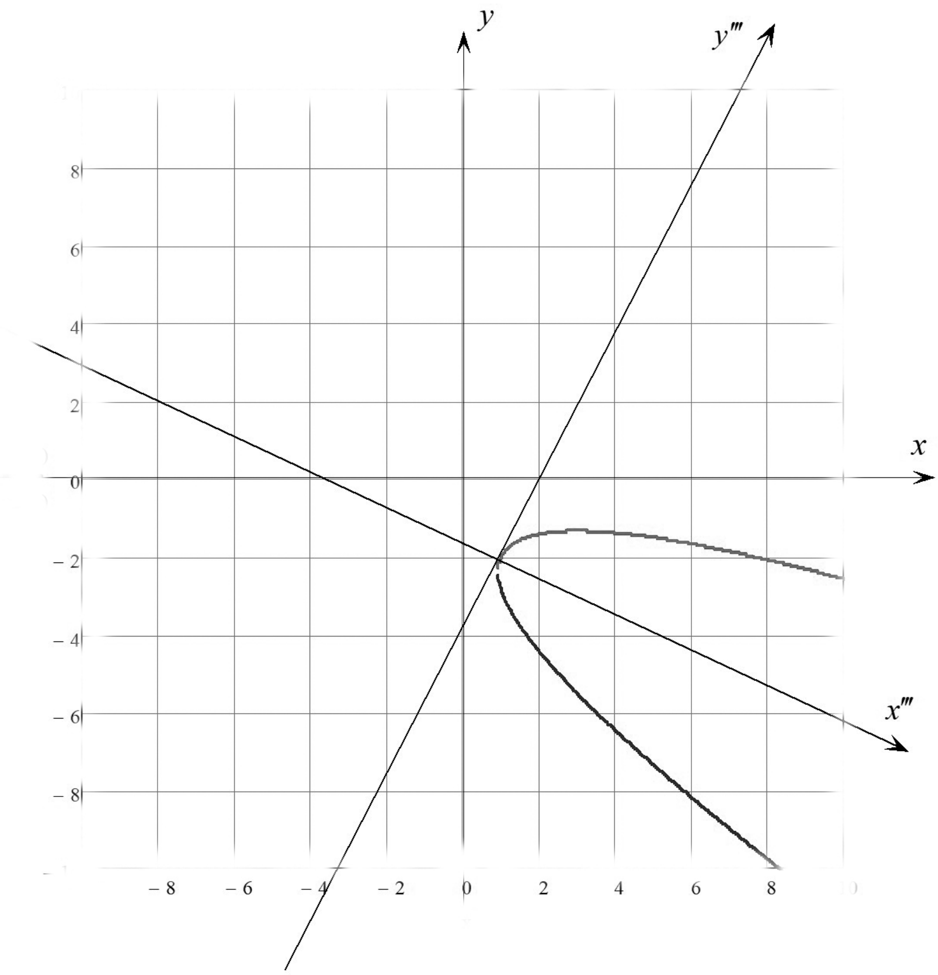
We obtain the canonical form of the equation of our line by renaming the axes and shifting the origin of coordinates, that is, by replacing: $\begin{cases} x'' = y''' - \frac{3}{\sqrt{5}}, \\ y'' = x''' + \frac{4}{\sqrt{5}}. \end{cases}$. **(C)**

This form will look like $y'''^2 = 2px'''$, that is, it is a parabola, for which the focal parameter $p = \frac{\sqrt{5}}{3}$.

4°. We find the final transition formulas by substituting (**C**) into (**B**) , and we substitute the result of this substitution into (**A**). We finally obtain $\begin{cases} x = \dfrac{2}{\sqrt{5}}x''' + \dfrac{1}{\sqrt{5}}y''' + 1, \\ y = -\dfrac{1}{\sqrt{5}}x''' + \dfrac{2}{\sqrt{5}}y''' - 2\,. \end{cases}$

This means that the final transition matrix $\|S\| = \begin{Vmatrix} \dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \\ -\dfrac{1}{\sqrt{5}} & \dfrac{2}{\sqrt{5}} \end{Vmatrix}$, and the final origin (the vertex of the parabola) will be at the point $\begin{Vmatrix} 1 \\ -2 \end{Vmatrix}$.

5°. Considering that the columns of the matrix $\|S\|$ are the coordinate columns of the basis vectors $\vec{e}_1\,''', \vec{e}_2\,'''$, and the vector $\vec{e}_1\,'''$ is the direction vector of the parabola axis, we construct a sketch:



A graphical solution to the problem is obtained by removing ('erasing') the axes of the canonical coordinate system $\{O''', x''', y'''\}$ from the drawing.