Lines and surfaces on the plane and in space (Theory)

Let a coordinate system in the plane $\{O, \vec{g}_1, \vec{g}_2\}$ and a numerical set Ω be given that is an interval (possibly infinite).

We will say that a line L in the plane is defined *parametrically* by a vector function $\vec{r} = \vec{F}(\tau)$ (or in coordinate form

$$\left\| \vec{r} \right\|_g = \left\| \begin{matrix} F_x(\tau) \\ F_y(\tau) \end{matrix} \right\|,$$

where $F_x(\tau), F_y(\tau)$ are continuous, scalar functions of argument τ, defined for $\tau \in \Omega$), if

1) for any $\tau \in \Omega$ point $\vec{r} = \vec{F}(\tau)$ lies in L;
2) for any point $\vec{r}_0$ lying on L, there exists $\tau_0 \in \Omega$ such that the equality holds $\vec{r}_0 = \vec{F}(\tau_0)$.

Sometimes a line in a plane is defined as an equation $G(x,y)=0$, which is obtained by eliminating the parameter τ from the system of equations $\begin{cases} x = F_x(\tau) \\ y = F_y(\tau) \end{cases}$, $\tau \in \Omega$.

1°. A straight line, for example, is defined by a vector function $\vec{r} = \vec{r}_0 + \tau \vec{a}$, where $\vec{a}$ is the direction vector, and $\vec{r}_0$ is one of the points of this line. The scalar form of defining a line in this case has the form

$$\begin{cases} x = x_0 + \tau a_x, \\ y = y_0 + \tau a_y, \end{cases} \quad \tau \in (-\infty, +\infty),$$

that is, $\begin{cases} F_x(\tau) = x_0 + \tau a_x, \\ F_y(\tau) = y_0 + \tau a_y, \end{cases} \quad \tau \in (-\infty, +\infty)$,

or $Ax + By + C = 0, \quad |A| + |B| > 0$, where $G(x,y) = Ax + By + C$.

2°. In a Cartesian coordinate system with an *orthonormal* basis, a circle of radius R with center at a point $\begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix}$ in parametric form can be defined as

$$\begin{cases} x = x_0 + R\cos\tau, \\ y = y_0 + R\sin\tau, \end{cases} \quad \tau \in [0,2\pi),$$

that is,

$$\begin{cases} F_x(\tau) = x_0 + R\cos\tau, \\ F_y(\tau) = y_0 + R\sin\tau, \end{cases} \quad \tau \in [0,2\pi),$$

or by the equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2,$$

$$\text{where} \quad G(x,y) = (x - x_0)^2 + (y - y_0)^2 - R^2.$$

A line L is called *algebraic* if its equation in a Cartesian coordinate system has the form $\sum_{k=0}^{m} \alpha_k x^{p_k} y^{q_k} = 0$, where p_k and q_k are non-negative integers, and the numbers α_k are not equal to zero simultaneously.

The number $N = \max_{k=[0,m]} \{p_k + q_k\}$ is called the *order of the algebraic equation*, where the maximum is found over all k, for which $\alpha_k \neq 0$. The *smallest* of the orders of the algebraic equations defining a given algebraic line is called the *order of the algebraic line*.

Name	**Equation**	**Order**
Straight line	$x + 3y + 2 = 0$	$(N = 1)$
Square parabola	$y - x^2 = 0$	$(N = 2)$
Hyperbola	$xy - 1 = 0$	$(N = 2)$
"*Cartesian leaf*"	$x^3 + y^3 - xy = 0$	$(N = 3)$

Theorem **The order of an algebraic line does not depend on the choice of coordinate system.**

Proof.

Let an algebraic line L have an equation $G(x,y)=0$ and order N in the coordinate system $\{O,\vec{g}_1,\vec{g}_2\}$. Let us move to the coordinate system $\{O,\vec{g}'_1,\vec{g}'_2\}$. The transition formulas have the form

$$\begin{cases} x=\sigma_{11}x'+\sigma_{12}y'+\beta_1, \\ y=\sigma_{21}x'+\sigma_{22}y'+\beta_2, \end{cases}$$

the equation of the line L in the “new” coordinate system will be

$$G(\sigma_{11}x'+\sigma_{12}y'+\beta_1,\ \sigma_{21}x'+\sigma_{22}y'+\beta_2)=0.$$

It follows from this that $N \ge N'$, that is, when moving to the “new” coordinate system, the order of the algebraic curve cannot increase.

Using similar reasoning for the reverse transition from the coordinate system $\{O,\vec{g}'_1,\vec{g}'_2\}$ to the system $\{O,\vec{g}_1,\vec{g}_2\}$, we obtain $N \le N'$ and finally $N = N'$.

Theorem is proven.

Figures in the plane can be defined using inequality-type constraints.

1°. In an *orthonormal* coordinate system, a set of conditions $\begin{cases} x \geq 0, \\ y \geq 0, \\ x + y - 5 \leq 0 \end{cases}$ defines a right isosceles triangle whose legs lie in the coordinate axes and have lengths of 5.

2°. In an *orthonormal* coordinate system, an inequality of the type $x^2 + y^2 - 4 \leq 0$ defines a circle of radius 2 with center at the origin.

Lines in space

Let a Cartesian coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ be given.

We will say that a line L in space is defined parametrically by a vector function $\vec{r} = \vec{F}(\tau)$ (or in coordinate form

$$\begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} F_x(\tau) \\ F_y(\tau) \\ F_z(\tau) \end{Vmatrix},$$

where $F_x(\tau), F_y(\tau), F_z(\tau)$ are continuous, scalar functions of τ, defined for $\tau \in \Omega$), if

1) for any $\tau \in \Omega$ point $\vec{r} = \vec{F}(\tau)$ lies in L,
2) for any point $\vec{r}_0$ lying in L, there exists $\tau_0 \in \Omega$, such that the equality is satisfied $\vec{r}_0 = \vec{F}(\tau_0)$.

Sometimes a line in space is defined by a system of equations

$$\begin{cases} G(x,y,z) = 0, \\ H(x,y,z) = 0, \end{cases}$$

which is obtained by excluding the parameter τ from the relations

$$\begin{cases} x = F_x(\tau), \\ y = F_y(\tau), \\ z = F_z(\tau), \end{cases} \quad \tau \in \Omega,$$

or by an equivalent equation, for example, of the form

$$G^2(x,y,z) + H^2(x,y,z) = 0.$$

1°. In a Cartesian coordinate system, a second-order algebraic line $x^2 + y^2 = 0 \ \forall z$ is a *straight line*.

2°. In an *orthonormal* coordinate system, a helical line of radius R with a pitch $2\pi a$ can be specified in the following parametric form:

$$\begin{cases} x = R\cos\tau, \\ y = R\sin\tau,\ \tau \in (-\infty, +\infty), \\ z = a\tau \end{cases} \qquad \text{or} \qquad \begin{cases} x = R\cos\dfrac{z}{a}, \\ y = R\sin\dfrac{z}{a}. \end{cases}$$

Surfaces in space

Let there be a Cartesian coordinate system $\{O,\vec{g}_1,\vec{g}_2,\vec{g}_3\}$ and Ω is a set of ordered pairs of numbers φ,θ, defined by the conditions: $\alpha \le \varphi \le \beta,\ \gamma \le \theta \le \delta$.

We will say that in space a surface S is defined parametrically by a vector function $\vec{r} = \vec{F}(\varphi,\theta)$ (or in coordinate form

$$,\left\|\vec{r}\right\|_g = \left\|\begin{matrix} F_x(\varphi,\theta) \\ F_y(\varphi,\theta) \\ F_z(\varphi,\theta) \end{matrix}\right\|$$

where $F_x(\varphi,\theta), F_y(\varphi,\theta), F_z(\varphi,\theta)$ are continuous scalar functions of two arguments φ,θ, defined for $\varphi,\theta \in \Omega$), if

1) for any ordered pair of numbers $\varphi,\theta \in \Omega$ the point $\vec{r} = \vec{F}(\varphi,\theta)$ lies in S,
2) for any $\vec{r}_0$ point lying in S, there exists an ordered pair of numbers $\varphi_0,\theta_0 \in \Omega$, such that the equality $\vec{r}_0 = \vec{F}(\varphi_0,\theta_0)$ holds.

Иногда поверхность в пространстве задается в виде уравнения $G(x,y,z)=0$, которое получается исключением φ и θ из системы уравнений Sometimes a surface in space is defined in the form of an equation $G(x,y,z)=0$, which is obtained by excluding φ and θ from the system of equations

$$\begin{cases} x = F_x(\varphi,\theta), \\ y = F_y(\varphi,\theta), \\ z = F_z(\varphi,\theta). \end{cases} \quad \varphi,\theta \in \Omega .$$

In an *orthonormal* coordinate system, a *sphere* of radius R with center at a point $\begin{Vmatrix} x_0 \\ y_0 \\ z_0 \end{Vmatrix}$ can be *parametrically* defined as

$$\begin{cases} x = x_0 + R\cos\varphi\sin\theta, \\ y = y_0 + R\sin\varphi\sin\theta, \\ z = z_0 + R\cos\theta, \end{cases} \quad \begin{array}{c} 0 \le \varphi < 2\pi, \\ 0 \le \theta \le \pi, \end{array}$$

and its equation in coordinates

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 .$$

A surface S is called *algebraic* if its equation in a Cartesian coordinate system has the form $\sum_{k=0}^{m} \alpha_k x^{p_k} y^{q_k} z^{r_k} = 0$, where p_k, q_k and r_k are non-negative integers, and the numbers α_k are not equal to zero simultaneously.

The number $N = \max_{k=[0,m]} \{p_k + q_k + r_k\}$ is called the *order of the algebraic equation*, where the maximum is found over all k for which $\alpha_k \neq 0$. The *smallest* of the orders of the algebraic equations defining a given algebraic surface is called the *order of the algebraic surface.*

Name	Equation	Order
Right circular cylinder	$x^2 + y^2 - 1 = 0$	$(N = 2)$
Sphere	$x^2 + y^2 + z^2 - R^2 = 0$	$(N = 2)$

Theorem **The order of an algebraic surface does not depend on the choice of coordinate system.**