STRAIGHT LINE AND PLANE IN SPACE

Forms of defining a plane in space

Let a coordinate system in space $\{O,\vec{g}_1,\vec{g}_2,\vec{g}_3\}$ and a plane S passing through a point $\vec{r}_0=\begin{Vmatrix} x_0 \\ y_0 \\ z_0 \end{Vmatrix}$ be given, with non-collinear vectors $\vec{p}=\begin{Vmatrix} p_x \\ p_y \\ p_z \end{Vmatrix}$ and $\vec{q}=\begin{Vmatrix} q_x \\ q_y \\ q_z \end{Vmatrix}$ lying on S. Vectors $\vec{p}=\begin{Vmatrix} p_x \\ p_y \\ p_z \end{Vmatrix}$ and $\vec{q}=\begin{Vmatrix} q_x \\ q_y \\ q_z \end{Vmatrix}$ are called *direction vectors* for S. Then the following statement is true:

- The set of position vectors of points of plane S can be represented as

$$\boxed{\vec{r}=\vec{r}_0+\varphi\,\vec{p}+\theta\,\vec{q}\,,}$$

where φ and θ are arbitrary real numbers.

Since this equation is equivalent to the condition of coplanarity of vectors $\vec{r}-\vec{r}_0$, $\vec{p}$ and $\vec{q}$, it can be written as

$$\boxed{(\vec{r}-\vec{r}_0,\vec{p},\vec{q}) = 0.}$$

Or, in coordinate $\det\begin{Vmatrix} x-x_0 & y-y_0 & z-z_0 \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{Vmatrix} = 0$.

- Any plane in any Cartesian coordinate system can be defined by an equation of the form $Ax + By + Cz + D = 0$, $|A| + |B| + |C| > 0$.

- Each equation of the form $Ax + By + Cz + D = 0$, $|A| + |B| + |C| > 0$ in any Cartesian coordinate system defines a certain plane.

The equation of a plane S passing through a point with a position vector $\vec{r}_0 = \begin{Vmatrix} x_0 \\ y_0 \\ z_0 \end{Vmatrix}$ perpendicular to a nonzero vector $\vec{n} = \begin{Vmatrix} n_x \\ n_y \\ n_z \end{Vmatrix}$ has the form

$$\boxed{(\vec{n}, \vec{r} - \vec{r}_0) = 0, \quad \text{or} \quad (\vec{n}, \vec{r}) = d, \quad \text{where} \quad d = (\vec{n}, \vec{r}_0).}$$

The vector $\vec{n}$ is called the *normal vector* of the plane S.

- If a plane S is defined in an *orthonormal* coordinate system $\{O,\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ by the equation $Ax+By+Cz+D=0$, $|A|+|B|+|C|>0$, then the vector $\vec{n}=\begin{Vmatrix} A \\ B \\ C \end{Vmatrix}$ is orthogonal to this plane S.

The equation of a plane passing through three pairwise non-coinciding and non-collinear points $\vec{r}_1=\begin{Vmatrix} x_1 \\ y_1 \\ z_1 \end{Vmatrix}$; $\vec{r}_2=\begin{Vmatrix} x_2 \\ y_2 \\ z_2 \end{Vmatrix}$; $\vec{r}_3=\begin{Vmatrix} x_3 \\ y_3 \\ z_3 \end{Vmatrix}$ has the form $(\vec{r}-\vec{r}_1,\vec{r}_2-\vec{r}_1,\vec{r}_3-\vec{r}_1)=0$.

Or, in coordinate representation

$$\det\begin{Vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{Vmatrix}=0..$$

Forms of defining a straight line in space

A straight line L in space, having a non-zero direction vector $\vec{a} = \left\| \begin{matrix} a_x \\ a_y \\ a_z \end{matrix} \right\|$ and passing through a point with position vector $\vec{r}_0 = \left\| \begin{matrix} x_0 \\ y_0 \\ z_0 \end{matrix} \right\|$, is defined by an equation of the form

$$\boxed{\vec{r} = \vec{r}_0 + \tau \vec{a}} \, .$$

If the parameter τ is excluded from the scalar notation of this equation $\begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y \\ z = z_0 + \tau a_z \end{cases}$, then we obtain a *standard* system of the form

$$\boxed{\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z}.}$$

A straight line L in space can be defined as the intersection line of planes of the form $(\vec{n}_1,\vec{r}) = d_1$ and $(\vec{n}_2,\vec{r}) = d_2$. Here $\vec{n}_1$ and $\vec{n}_2$ are *non-collinear* normal vectors of these planes, and d_1 and d_2 are some numbers. The vector description of the straight line L will be

$$\boxed{\begin{cases}(\vec{n}_1,\vec{r}) = d_1\\(\vec{n}_2,\vec{r}) = d_2\end{cases}}$$

If the position vector $\vec{r}_0$ of some point of a straight line L is known, then its description will be $\begin{cases}(\vec{n}_1,\vec{r}-\vec{r}_0) = 0,\\(\vec{n}_2,\vec{r}-\vec{r}_0) = 0.\end{cases}$ The coordinate form of the description of L in these cases will be

$$\begin{cases}A_1x + B_1y + C_1z + D_1 = 0,\\A_2x + B_2y + C_2z + D_2 = 0.\end{cases}$$

The condition of collinearity of vectors $\vec{a}$ and $\vec{r}-\vec{r}_0$ when specifying a straight line L can be written as $[\vec{a},\vec{r}-\vec{r}_0]=\vec{o}$ or

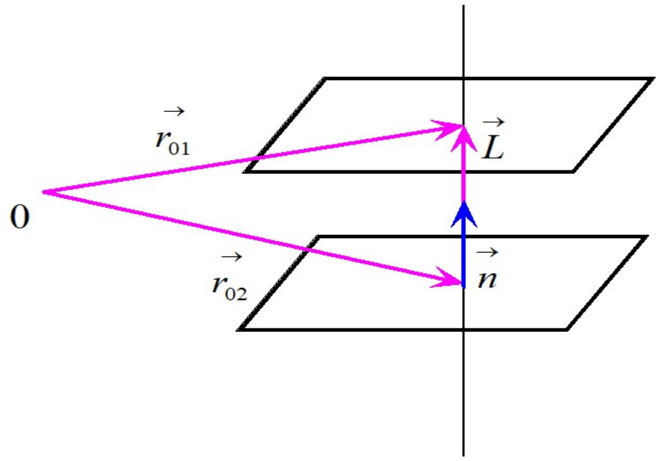
$$\boxed{[\vec{a},\vec{r}]=\vec{b}\,,}$$

where $\vec{b}=[\vec{a},\vec{r}_0]$.

In an Cartesian coordinate system $\{O,\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ with the *orthonormal basis*, this method of describing a straight line in space L has the form

$$\det\begin{Vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_x & a_y & a_z \\ x & y & z \end{Vmatrix}=\vec{b} \qquad \text{or} \qquad \begin{cases} a_y z - a_z y = b_x, \\ a_z x - a_x z = b_y, \\ a_x y - a_y x = b_z. \end{cases}$$

Note that in the latter system only two equations of the three are independent.



Task 6.1. *Find the distance between planes* $(\vec{n},\vec{r}) = d_1$ *and* $(\vec{n},\vec{r}) = d_2$.

Solution: 1) Let $\vec{r}_{01}$ and $\vec{r}_{02}$ be the position vectors of the intersection points of the common perpendicular with the planes. Then we have $\vec{L} = \vec{r}_{01} - \vec{r}_{02}$ and $\vec{L} = \lambda\vec{n}$. The desired distance is obviously equaled to $S = \left|\vec{L}\right|$.

2) We can find λ from the condition $\vec{r}_{01} - \vec{r}_{02} = \lambda\vec{n}$. Multiplying both parts scalarly by $\vec{n}$, we obtain

$$(\vec{r}_{01},\vec{n}) - (\vec{r}_{02},\vec{n}) = \lambda(\vec{n},\vec{n}) \quad\Rightarrow\quad d_1 - d_2 = \lambda\left|\vec{n}\right|^2 \quad\Rightarrow\quad \lambda = \frac{d_1 - d_2}{\left|\vec{n}\right|^2}.$$

3) Whence $S = \left|\vec{L}\right| = \left|\lambda\vec{n}\right| = |\lambda|\left|\vec{n}\right| = \dfrac{|d_1 - d_2|}{\left|\vec{n}\right|}$

Solution is found

Task 6.2. *Write an equation of a plane passing through a point* $A(5,-4,3)$, *perpendicular to a line passing through points* $B(-1,-2,1)$ *and* $C(-6,-4,3)$. *The coordinate system is Cartesian with an orthonormal basis .*

Solution: 1) Let the points A, B, C have position vectors $\vec{r}_0, \vec{r}_1$ and $\vec{r}_2$, respectively, and an arbitrary point of the plane has position vector $\vec{r}$.

2) Note that the vectors $\vec{r}_2 - \vec{r}_1$ and $\vec{r} - \vec{r}_0$ in this case are orthogonal for any $\vec{r}$. Therefore, the desired equation will be $(\vec{r}_2 - \vec{r}_1, \vec{r} - \vec{r}_0) = 0$.

3) Let the vector $\vec{r}$ has coordinates $\left\| \begin{matrix} x \\ y \\ z \end{matrix} \right\|$. Then the equation of the plane has the form

$$(-6+1)(x-5)+(-4+2)(y+4)+(3-1)(z-3)=0 \qquad \Rightarrow \qquad 5x+2y-2z=11$$

.

Solution is found

Task 6.3. *Find a plane passing through the line* $\begin{cases} x = 1 - 5\tau \\ y = 2 - \tau \\ z = 1 + 4\tau \end{cases}$, *parallel to the line* $\begin{cases} x = -3 + 3\tau \\ y = -2 - 5\tau \\ z = 2 - 4\tau \end{cases}$. *The Cartesian coordinate system is arbitrary.*

Solution: 1) Let the given lines have the following parametric form in vector form

$$\vec{r} = \vec{r}_{01} + \tau\,\vec{a}_1 \quad \text{and} \quad \vec{r} = \vec{r}_{02} + \tau\,\vec{a}_2\,.$$

That is, we know the point $\vec{r}_{01}$, through which the sought plane is guaranteed to pass, since the first line also passes through this point.

2) Vectors $\vec{a}_1$ and $\vec{a}_2$ are colinear with the desired plane by condition. Let the vector $\vec{r}$ of an arbitrary point of this plane has coordinates $\left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\|$. This means that the triple of vectors $\vec{r}-\vec{r}_{01},\ \vec{a}_1$ and $\vec{a}_2$ is coplanar with the desired plane, and its equation can be written as $(\vec{r}-\vec{r}_{01},\ \vec{a}_1,\ \vec{a}_2)=0$.

3) In an arbitrary coordinate system, this equation will be

$$\det\left\|\begin{matrix} x-1 & y-2 & z-1 \\ -5 & -1 & 4 \\ 3 & -5 & -4 \end{matrix}\right\|\cdot(g_1,g_2,g_3)=0,$$

where the basis vectors $\{g_1;g_2;g_3\}$ are linearly independent and, therefore, their mixed product is nonzero. Then the equation of the desired plane will be

$$\det\left\|\begin{matrix} x-1 & y-2 & z-1 \\ -5 & -1 & 4 \\ 3 & -5 & -4 \end{matrix}\right\|=0 \quad\Rightarrow\quad 6x-2y+7z=9.$$

Solution is found

Task 6.4. *Find the intersection point of a line* $\begin{cases} x = -3 + 4\tau \\ y = 1 - 4\tau \\ z = -5 + \tau \end{cases}$ *and a plane* $x + 4z = -7$.

The Cartesian coordinate system is arbitrary.

Solution: 1) Let the given line and plane have the following form in vector form $\vec{r} = \vec{r}_0 + \tau\,\vec{a}$ and $(\vec{n}, \vec{r}) = d$. Let us denote the desired intersection point as $\vec{R}$, and we can assume that $\vec{R} = \vec{r}_0 + \lambda\,\vec{a}$.

2) Since $\vec{R}$ belongs to both the line and the plane, the value of the parameter λ can be found from the system of equations

$$\begin{cases} \vec{R} = \vec{r}_0 + \lambda\,\vec{a} \\ (\vec{n}, \vec{R}) = d \end{cases} \Rightarrow \quad (\vec{n}, \vec{r}_0 + \lambda\,\vec{a}) = d \quad \Rightarrow \quad \lambda = \frac{d - (\vec{n}, \vec{r}_0)}{(\vec{n}, \vec{a})}.$$

Where from $\vec{R} = \vec{r}_0 + \dfrac{d - (\vec{n}, \vec{r}_0)}{(\vec{n}, \vec{a})}\vec{a}$.

3) It is more convenient to find the values of the coordinates of the desired point $\vec{R} = \begin{Vmatrix} X \\ Y \\ Z \end{Vmatrix}$ not using the obtained formula, but directly from the system of linear equations

$$\begin{cases} X = -3 + 4\lambda, \\ Y = 1 - 4\lambda, \\ Z = -5 + \lambda, \\ X + 4Z + 7 = 0. \end{cases}$$

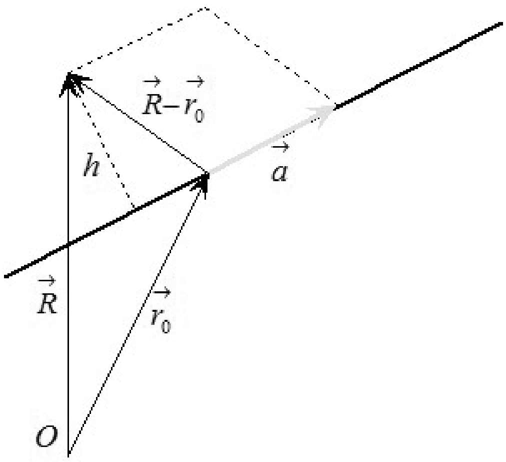
If we substitute X and Z from the first and third equations into the fourth, then we immediately get that $\lambda = 2$ and, therefore, $\vec{R} = \begin{Vmatrix} 5 \\ -7 \\ -3 \end{Vmatrix}$.

Solution is found

Task 6.5 *Find the distance from a point with a position vector* $\vec{R}$ *to a line* $\vec{r} = \vec{r}_0 + \tau \vec{a}$.

Solution: The distance h in space from a certain point with a position vector $\vec{R}$ to a line $\vec{r} = \vec{r}_0 + \tau \vec{a}$ can be found using the following property. The area S of a parallelogram constructed on a pair of vectors is equal to the length of the vector product of these vectors. As a result, we obtain

$$h = \frac{S}{\left|\vec{a}\right|} = \frac{\left|[\vec{R} - \vec{r}_0, \vec{a}]\right|}{\left|\vec{a}\right|} .$$

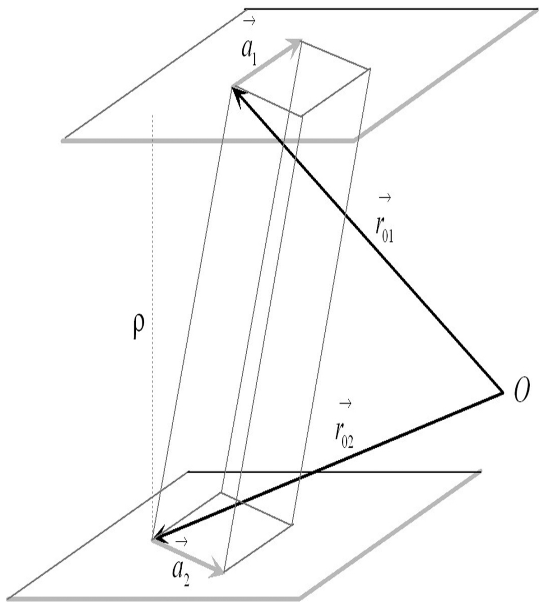


Solution is found

Task 6.6 *Find the distance between the lines* $\vec{r} = \vec{r}_{01} + \tau\,\vec{a}_1$ *and* $\vec{r} = \vec{r}_{02} + \tau\,\vec{a}_2$.

Solution: 1°. If the vectors $\vec{a}_1$ and $\vec{a}_2$ are collinear, then the solution is similar to the solution of the task 6.5.

2°. Let the vectors $\vec{a}_1$ and $\vec{a}_2$ be non-collinear, then we construct a pair of planes parallel to these vectors, one of which contains the point $\vec{r}_{01}$, and the other the point $\vec{r}_{02}$.



The volume of the parallelepiped constructed on the vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{r}_{02} - \vec{r}_{01}$, is equal, on the one hand, to the product of the area of the parallelogram located at the base, by the desired value ρ and $\left|(\vec{r}_{02} - \vec{r}_{01}, \vec{a}_1, \vec{a}_2)\right|$ – on the other. Whence we find that

$$\rho = \frac{|(\vec{r}_{02} - \vec{r}_{01}, \vec{a}_1, \vec{a}_2)|}{|[\vec{a}_1, \vec{a}_2]|}.$$

Solution is found

Task 6.7 *A plane* $(\vec{n},\vec{r})=d$ *and a line* $[\vec{a},\vec{r}]=\vec{b}$ *are given. Find the position vector of their intersection point if* $(\vec{n},\vec{a})\neq 0$.

Solution: 1°. Multiplying both sides of the vector equation of the line from the left by $\vec{n}$, we obtain $[\vec{n},[\vec{a},\vec{r}]]=[\vec{n},\vec{b}]$. Substituting the desired vector $\vec{R}$ into this relation and applying the "*bac-cab*" formula, we arrive at the equality

$$\vec{a}(\vec{n},\vec{R})-\vec{R}(\vec{n},\vec{a})=[\vec{n},\vec{b}].$$

Since the point $\vec{R}$ belongs to the given plane, the equality $(\vec{n},\vec{R})=d$ is true. Then, under the constraint $(\vec{n},\vec{a})\neq 0$, we obtain

$$\vec{R}=\frac{d\,\vec{a}-[\vec{n},\vec{b}]}{(\vec{n},\vec{a})}.$$

Solution is found