STRAIGHT LINE IN A PLANE

1°. Forms of defining a straight line in a plane

Let a coordinate system in a plane $\{O, \vec{g}_1, \vec{g}_2\}$ and a straight line L passing through a point $\vec{r}_0 = \begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix}$ be given. A nonzero vector $\vec{a} = \begin{Vmatrix} a_x \\ a_y \end{Vmatrix}$ lying in L, is called the *direction vector* for L. Then the following statement is true.

> The set of position vectors of points in a straight line L can be represented as $\vec{r} = \vec{r}_0 + \tau \vec{a}$, where τ is an arbitrary real parameter.

Let's find the coordinate form for $\vec{r} = \vec{r}_0 + \tau\vec{a}$ in the chosen coordinate system.

We have: from $\vec{r} = \vec{r}_0 + \tau\vec{a}$ it follows $\left\| \vec{r} \right\|_g = \left\| \vec{r}_0 + \tau\vec{a} \right\|_g$.

Using the statements of the theorems on actions with vectors in coordinates, we obtain

$$\left\| \vec{r} \right\|_g = \left\| \vec{r}_0 \right\|_g + \left\| \tau\vec{a} \right\|_g \quad \text{and} \quad \left\| \vec{r} \right\|_g = \left\| \vec{r}_0 \right\|_g + \tau\left\| \vec{a} \right\|_g .$$

Let in the chosen coordinate system $\left\| \vec{r} \right\|_g = \begin{Vmatrix} x \\ y \end{Vmatrix}$, $\left\| \vec{r}_0 \right\|_g = \begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix}$ and $\left\| \vec{a} \right\|_g = \begin{Vmatrix} a_x \\ a_y \end{Vmatrix}$, then the equation of the line in *matrix* form will be $\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix} + \tau \begin{Vmatrix} a_x \\ a_y \end{Vmatrix}$.

Using operations with *matrices*, we reduce the resulting equation to the form

$$\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix} + \begin{Vmatrix} \tau a_x \\ \tau a_y \end{Vmatrix} \quad \Rightarrow \quad \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} x_0 + \tau a_x \\ y_0 + \tau a_y \end{Vmatrix} .$$

Finally, by the definition of equality of matrices, we obtain $\begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y \end{cases} \quad \forall \tau \in \mathbf{R}$.

In coordinate representation it follows from $\begin{cases} x = x_0 + \tau a_x \\ y = y_0 + \tau a_y \end{cases}$

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} \ ; \quad a_x a_y \neq 0$$

$$y = y_0 \ ; \ \forall x, \ \text{if} \ a_y = 0$$

$$x = x_0 \ ; \ \forall y, \ \text{if} \ a_x = 0,$$

This notation is called the *symmetrical* form of the equation of a straight line in a plane.

We obtain that the following statements are true

- Any line in any Cartesian coordinate system can be defined by an equation of the form $Ax + By + C = 0$, $|A| + |B| > 0$.

- Each equation of the form $Ax + By + C = 0$, $|A| + |B| > 0$ in any Cartesian coordinate system is an equation of some line.

- In order for two equations of the form $A_1x + B_1y + C_1 = 0$, $|A_1| + |B_1| > 0$ and $A_2x + B_2y + C_2 = 0$, $|A_2| + |B_2| > 0$ to be equations of the same line, it is necessary and sufficient that there exists a number $\lambda \neq 0$ such that

$$A_1 = \lambda A_2 \ ; \ B_1 = \lambda B_2 \ ; \ C_1 = \lambda C_2 \ .$$

Task 5.01 *What set of points in the coordinate system* $\{O, \vec{g}_1, \vec{g}_2\}$ *in the plane does the equation* $Ax + By + C = 0$ *describe*?

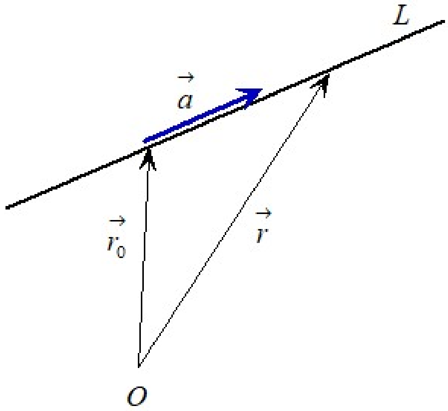
Solution

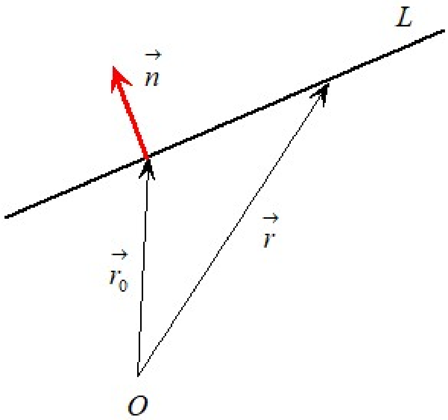
1°. If $|A| + |B| > 0$ and $C \in \mathbf{R}$, then this set is some *line*.

2°. If $A = 0, B = 0$ and $C \neq 0$, then the set is the *empty set*.

3°. If $A = 0, B = 0$ and $C = 0$, then the set is the *entire coordinate plane*.

Solution is found





The equation of a straight line L passing through a given point $\vec{r}_0 = \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\|$, perpendicular to a non-zero vector $\vec{n} = \left\| \begin{matrix} n_x \\ n_y \end{matrix} \right\|$, has the form $(\vec{n}, \vec{r} - \vec{r}_0) = 0$.

Or $(\vec{n}, \vec{r}) = d$, where $d = (\vec{n}, \vec{r}_0)$.

Vtctor $\vec{n}$ is called the *normal* vector for L

Task 5.02 *Write the equation* $(\vec{n},\vec{r}) = d$ *of a straight line L as* $(\vec{n},\vec{r}-\vec{r}_0) = 0$.

Solution

Since the vector $\vec{n}$ is contained in both forms of the equation, we only need to find the vector $\vec{r}_0$.

As $\vec{r}_0$ we can take the position vector of any point for the line *L*. Therefore, we take as $\vec{r}_0$ the *vector of the orthogonal projection of the origin* onto the straight line *L*, which exists for any straight line. In this case, the equality $\vec{r}_0 = \lambda \vec{n}$ is obviously satisfied, where λ is some number.

Since the point $\vec{r}_0$ belongs to the straight line *L*, we have

$$(\vec{n},\lambda \vec{n}) = d \qquad \Rightarrow \qquad \lambda = \frac{d}{\left|\vec{n}\right|^2} .$$

Solution is found

By comparing different forms of the straight line:

$$\frac{x-x_0}{a_x}=\frac{y-y_0}{a_y} \ , \ (\vec{n},\vec{r})=d \ , \ (\vec{n},\vec{r}-\vec{r}_0)=0 \ \text{ и } \ Ax+By+C=0 \ , \ |A|+|B|>0 \ ,$$

we can establish the geometric meaning of the coefficients A, B, C.

In any Cartesian coordinate system $A=a_y$, $B=-a_x$. In a Cartesian coordinate system with an orthonormal basis $A=n_x$, $B=n_y$.

The equation of a straight line passing through two non-coinciding points $\vec{r}_1 = \left\| \begin{matrix} x_1 \\ y_1 \end{matrix} \right\|$ and $\vec{r}_2 = \left\| \begin{matrix} x_2 \\ y_2 \end{matrix} \right\|$ in vector form will look like $\vec{r} = (1-\tau)\vec{r}_1 + \tau \vec{r}_2$.

In coordinate representation (after eliminating τ):

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}, \quad \text{if } (x_2 - x_1)(y_2 - y_1) \neq 0$$

$$y = y_1; \ \forall x, \quad \text{if } y_2 = y_1$$

$$x = x_1; \ \forall y, \quad \text{if } x_2 = x_1.$$

A *linear inequality* $Ax + By + C \ge 0$, $|A| + |B| > 0$ defines a part of the coordinate plane whose points satisfy the inequality. In this case the straight line $Ax + By + C = 0$, $|A| + |B| > 0$ is a fragment of the boundary of this part of the plane.

Task 5.03 *Construct a coordinate description of a right triangle ABC whose right angle vertex C is at the origin of the orthonomial coordinate system* $\{Oxy\}$, *and the other two vertices have position vectors* $\vec{r}_A = \begin{Vmatrix} 3 \\ 0 \end{Vmatrix}$ *and* $\vec{r}_B = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}$, *respectively.*

Solution

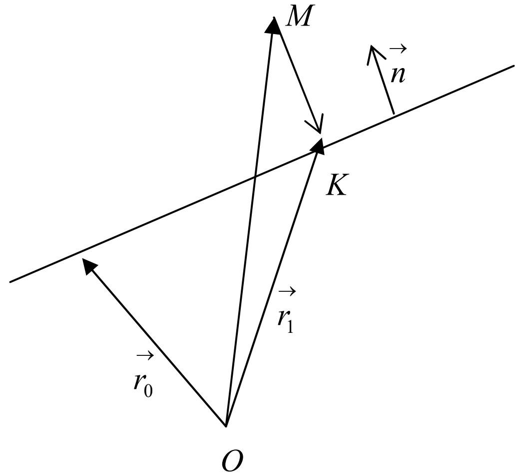
The desired description is the following system of non-strict linear inequalities

$$\begin{cases} x \ge 0, \\ y \ge 0, \\ x + 3y \le 3. \end{cases}$$

Solution is found

Task 5.04 *A coordinate system* $\{O,\vec{g}_1,\vec{g}_2\}$ *in a plane and a line L with the equation* $(\vec{n},\vec{r}-\vec{r}_0)=0$ *are given. Find the distance from this line to a point M whose position vector is* $\vec{r}_1=\left\|\begin{matrix}x_1\\y_1\end{matrix}\right\|$.

Solution



1°. Let $\vec{MK}=\lambda\vec{n}$, then $\vec{r}=\vec{r}_1+\lambda\vec{n}$.

2°. The point K belongs to this line, so the equality is true

$$(\vec{n},\vec{r}_1+\lambda\,\vec{n}-\vec{r}_0)=0\,.$$

Whence $\lambda=-\dfrac{(\vec{n},\vec{r}_1-\vec{r}_0)}{|\vec{n}|^2}$.

3°. Substituting λ into the formula for $\vec{MK}$, we obtain $|\vec{MK}|=\left|(\vec{r}_1-\vec{r}_0,\dfrac{\vec{n}}{|\vec{n}|})\right|$.

4°. Let the coordinate system be orthonormal. Then for the line

$$Ax+By+C=0\,,\qquad |A|+|B|>0$$

the vector $\vec{n}=\left\|\begin{matrix}A\\B\end{matrix}\right\|$ is normal. Therefore

$$|\vec{MK}|=\frac{|A(x_1-x_0)+B(y_1-y_0)|}{\sqrt{A^2+B^2}}\,.$$

Considering that the point $\vec{r}_0$ lies on the line *L*, we have $Ax_0+By_0+C=0$. Therefore the answer can be written as

$$|\vec{MK}|=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}\,.$$

Solution is found.

Task 5.05 *In a general Cartesian coordinate system, a line* $9x-5y-8=0$ *and points* $\vec{r}_A = \begin{Vmatrix} -8 \\ -9 \end{Vmatrix}$ *and* $\vec{r}_B = \begin{Vmatrix} -2 \\ -6 \end{Vmatrix}$ *are given. Determine analytically whether these points lie on the same or different sides of the line.*

Solution:

1) Find the value for the linear function $9x-5y-8$ at point A:
   we get $9(-8)-5(-9)-8=-35<0.$

2) Find the value for the linear function $9x-5y-8$ at point B:
   we get $9(-2)-5(-6)-8=4>0.$

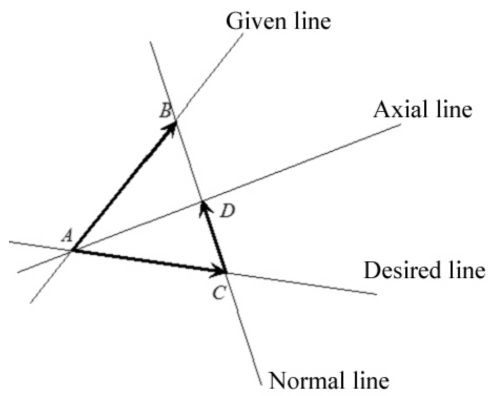
This means that points A and B lie on different sides of the line $9x-5y-8=0$.

Solution is found

Task 5.06 *On a plane with an orthonormal coordinate system, find the "desired" line, which is symmetrical to the "given" line with an equation* $11x - 2y + 31 = 0$*, relative to the "axial" line with the equation* $4x - 3y + 9 = 0$.

Solution:

1°. Let the *desired* line pass through two non-coinciding points with coordinates $(x_0; y_0)$ and $(x_1; y_1)$. Then its equation will be $\dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0}$.



2°. As the first point, we can choose point *A* − the intersection point of the *given* and *axial* lines. The coordinates of this point are obviously determined by the system of equations:

$$\begin{cases} 11x_0 - 2y_0 = -31, \\ 4x_0 - 3y_0 = -9, \end{cases}$$

which has a solution $\begin{cases} x_0 = -3, \\ y_0 = -1. \end{cases}$

3°. On the *given* line, we choose some point *B* that does not coincide with point *A*. For example, a point with coordinates $(-1;10)$. Let us draw a line through *B* *normal* (that is, perpendicular) to the *axial* line. Let point C, belonging to the *normal* line, be symmetrical to point *B* relative to the *axial* line,

In this case, if we designate the intersection point of the *axial* and *normal* lines as *D*, we will have $|BD| = |DC|$.

4°. Now let us find the coordinates of point C. First, let us note that, due to the choice of points B and C, the normal vector of the *axial* line $\vec{CD}$ will be the direction vector of the *normal* line. And, since the coordinate system is rectangular, the equation of the *normal* line will be $3x+4y+K=0$, where K is some constant.

We will find the value of K, taking into account that point B belongs to the *normal* line, that is,

$$3\cdot(-1)+4\cdot 10+K=0 \qquad \Rightarrow \qquad K=-37\,.$$

Therefore, the *normal* line has an equation $3x+4y-37=0$ and point C belongs to this line.

5°. On the other hand, let point C have coordinates $(x_1;y_1)$. Then, since $|BD|=|DC|$, using the formula for the *distance from a point to a line*, we write this equality as

$$\frac{|4\cdot(-1)-3\cdot 10+9|}{\sqrt{4^2+3^2}}=\frac{|4x_1-3y_1+9|}{\sqrt{4^2+3^2}} \quad \text{or} \quad \frac{|-25|}{5}=\frac{|4x_1-3y_1+9|}{5}.$$

Knowing that points B and C lie on different sides of the *axial* line, we open modules with opposite signs for their interiors, which gives $4x_1-3y_1-16=0$.

6°. Thus, for the coordinates of point C we have two conditions, which we write down in the form of a system $\begin{cases} 3x_1+4y_1=37, \\ 4x_1-3y_1=16, \end{cases}$ from which we obtain that $\begin{cases} x_1=7, \\ y_1=4. \end{cases}$

7°. Finally, substituting the values of the coordinates of points A and C into the formula from point 1°, we obtain the equation of the desired line:

$$\frac{x-(-3)}{7-(-3)}=\frac{y-(-1)}{4-(-1)} \quad \text{or, after simplification,} \quad x-2y+1=0\,.$$

Solution is found