Vector Products

1°. Scalar Product

The *scalar product* of nonzero vectors $\vec{a}$ and $\vec{b}$ is a number equal to the product of the lengths of these vectors by the cosine of the angle between them. If at least one of the factors is a zero vector, the scalar product is considered to be zero.

The scalar product is denoted as $(\vec{a},\vec{b})$. Then, by definition,

$$(\vec{a},\vec{b}) = |\vec{a}\,||\,\vec{b}\,|\cos\varphi\,;\; 0 \le \varphi \le \pi\,,$$

where φ is the angle between the factor vectors.

Properties of the scalar product

1°/ $(\vec{a},\vec{b}) = 0$ with $\vec{a} \neq \vec{o}$ and $\vec{b} \neq \vec{o}$? if $\vec{a}$ and $\vec{b}$ are mutually orthogonal,

2°. $(\vec{a},\vec{b}) = (\vec{b},\vec{a})$ (commutativity),

3°. $(\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2, \vec{b}) = \lambda_1 (\vec{a}_1,\vec{b}) + \lambda_2 (\vec{a}_2,\vec{b})$ (linearity)

4°. $(\vec{a},\vec{a}) = |\vec{a}|^2 \geq 0 \ \forall \vec{a}; \quad |\vec{a}| = \sqrt{(\vec{a},\vec{a})}$,

(conditions: $(\vec{a},\vec{a}) = 0$ and $\vec{a} = \vec{o}$ are equivalent),

5°. For $\vec{a} \neq \vec{o}$ and $\vec{b} \neq \vec{o}$ $\quad \cos\varphi = \dfrac{(\vec{a},\vec{b})}{|\vec{a}||\vec{b}|}$.

2°. **Vector product**

A *vector product* of non-collinear vectors $\vec{a}$ and $\vec{b}$ a vector $\vec{c}$ such that

1°. $|\vec{c}|=|\vec{a}||\vec{b}|\sin\varphi$, where is the angle between the vectors $\vec{a},\vec{b}$; $0<\varphi<\pi$.

2°. The vector $\vec{c}$ is orthogonal to the vector $\vec{a}$ and the vector $\vec{b}$.

3°. The triple of vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ is right-handed.

In the case where the factors are collinear, the vector product is considered equal to the zero vector.

The vector product is denoted as $[\vec{a},\vec{b}]$.

Properties of vector product

1°. $\left|[\vec{a},\vec{b}]\right|$ is equal to the area of the parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$.

2°. For nonzero vectors $\vec{a}$ and $\vec{b}$ to be collinear, it is necessary and sufficient that their vector product be equal to the zero vector.

3°. $[\vec{a},\vec{b}]=-[\vec{b},\vec{a}]$ (anticommutativity)

4°. $[\lambda\vec{a},\vec{b}]=\lambda[\vec{a},\vec{b}]$.

5°. $[\vec{a}+\vec{b},\vec{c}]=[\vec{a},\vec{c}]+[\vec{b},\vec{c}]$ (distributivity).

3°. Mixed product

The *mixed* (or *vector-scalar*) product of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, denoted as $(\vec{a}, \vec{b}, \vec{c})$, is the number $([\vec{a}, \vec{b}], \vec{c})$.

Properties of the mixed product

1°. The absolute value of the mixed product $(\vec{a}, \vec{b}, \vec{c})$ is equal to the volume of the parallelepiped constructed on the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. The sign of the mixed product is positive if the triple of $\vec{a}$, $\vec{b}$, $\vec{c}$ is right-handed, and negative if it is left-handed.

2°. $(\vec{a}, \vec{b}, \vec{c}) = (\vec{c}, \vec{a}, \vec{b}) = (\vec{b}, \vec{c}, \vec{a}) = -(\vec{b}, \vec{a}, \vec{c}) = -(\vec{c}, \vec{b}, \vec{a}) = -(\vec{a}, \vec{c}, \vec{b})$;

3°. $(\lambda \vec{a}, \vec{b}, \vec{c}) = \lambda(\vec{a}, \vec{b}, \vec{c})$;

4°. $(\vec{a}_1 + \vec{a}_2, \vec{b}, \vec{c}) = (\vec{a}_1, \vec{b}, \vec{c}) + (\vec{a}_2, \vec{b}, \vec{c})$,

The mixed product is equal to zero if there is at least one collinear pair among the factors.

4° Double vector product

The *double vector product* of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is called the vector $[\vec{a}\,[\vec{b}\;\vec{c}\,]]$.

Property of the double vector product

$$[\vec{a},[\vec{b},\vec{c}\,]]=\vec{b}(\vec{a},\vec{c}\,)-\vec{c}(\vec{a},\vec{b}\,),$$

Task 4.01 *What angle do vectors* $\vec{a}$ *and* $\vec{b}$ *form if it is known that* $\vec{a}+2\vec{b}$ *and* $5\vec{a}-4\vec{b}$ *are orthogonal?*

Solution

If vectors $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are orthogonal, then their scalar product is zero. Taking into account the commutativity of the scalar product and the conditions $|\vec{a}|=|\vec{b}|=1$ we have

$$0=(\vec{a}+2\vec{b},5\vec{a}-4\vec{b})=5(\vec{a},\vec{a})-4(\vec{a},\vec{b})+10(\vec{b},\vec{a})-8(\vec{b},\vec{b})=$$
$$=5|\vec{a}|^2+6(\vec{a},\vec{b})-8|\vec{b}|^2=6(\vec{a},\vec{b})-3.$$

Since $(\vec{a},\vec{b})=\dfrac{1}{2}$ and $\cos\varphi=\dfrac{1}{2} \quad\Rightarrow\quad \varphi=\dfrac{\pi}{3}$.

Solution is found.

Task .4.02 *Show that the vector product of a pair of vectors does not change if a vector collinear to the first factor is added to the second factor.*

Solution

Let $[\vec{a},\vec{b}]$ and $\vec{c}=\vec{b}+\lambda\,\vec{a}$ be given. For $[\vec{a},\vec{c}]$ we have

$$[\vec{a},\vec{c}]=[\vec{a},\vec{b}+\lambda\,\vec{a}]=[\vec{a},\vec{b}]+\lambda[\vec{a},\vec{a}]=[\vec{a},\vec{b}],$$

since $[\vec{a},\vec{a}]=\vec{o}$.

Solution is found

Note that we have also shown that it is impossible to uniquely indicate the second factor for a vector product and one of its factors.

Task 4.03

Find a vector $\vec{x}$ *lying in the plane of vectors* $\vec{a}$ *and* $\vec{b}$ *if*

$$\begin{cases}(\vec{a},\vec{x})=\alpha,\\(\vec{b},\vec{x})=\beta,\end{cases}$$

and vectors $\vec{a}$ *and* $\vec{b}$ *are non-collinear.*

Solution

Vectors $\vec{a}$ and $\vec{b}$ form a basis in their plane. Therefore, vector $\vec{x}$ can be (and uniquely) expanded in this basis

$$\vec{x}=\xi\vec{a}+\eta\vec{b}\quad.$$

We can find the expansion coefficients from the system of equations

$$\begin{cases}(\vec{a},\vec{a})\xi+(\vec{a},\vec{b})\eta=\alpha,\\(\vec{b},\vec{a})\xi+(\vec{b},\vec{b})\eta=\beta.\end{cases}$$

Solution is found.

Task 4.04

Find vector $\vec{x}$ *if*

$$\begin{cases} (\vec{a},\vec{x}) = \alpha, \\ (\vec{b},\vec{x}) = \beta, \\ (\vec{c},\vec{x}) = \gamma, \end{cases}$$

and the vectors $\vec{a}$, $\vec{b}$ *and* $\vec{c}$ *are not coplanar.*

Solution

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent, so the vectors $[\vec{a},\vec{b}]$, $[\vec{b},\vec{c}]$ and $[\vec{c},\vec{a}]$ are also linearly independent. Therefore, they form a basis in space and vector $\vec{x}$ can be (and uniquely) expanded in this basis

$$\vec{x} = \xi[\vec{a},\vec{b}] + \eta[\vec{b},\vec{c}] + \kappa[\vec{c},\vec{a}] \quad .$$

We can find the expansion coefficients from the system of equations

$$\begin{cases} (\vec{a},\vec{a},\vec{b},)\xi + (\vec{a},\vec{b},\vec{c},)\eta + (\vec{a},\vec{c},\vec{a},)\kappa = \alpha \ , \\ (\vec{b},\vec{a},\vec{b},)\xi + (\vec{b},\vec{b},\vec{c},)\eta + (\vec{b},\vec{c},\vec{a},)\kappa = \beta \ , \\ (\vec{c},\vec{a},\vec{b},)\xi + (\vec{c},\vec{b},\vec{c},)\eta + (\vec{c},\vec{c},\vec{a},)\kappa = \gamma \ , \end{cases}$$

which, by the properties of the mixed product, is equivalent to the system

$$\begin{cases} (\vec{a},\vec{b},\vec{c},)\eta = \alpha \ , \\ (\vec{b},\vec{c},\vec{a},)\kappa = \beta \ , \\ (\vec{c},\vec{a},\vec{b},)\xi = \gamma \ . \end{cases}$$

Solution is found.

Test 4.05 *Find all vectors* $\vec{x}$ *satisfying the relation*

$$[\vec{a},\vec{x}]+[\vec{x},\vec{b}]=[\vec{a},\vec{b}],$$

if vectors $\vec{a}$ *and* $\vec{b}$ *are non-collinear.*

Решение

We multiply both sides of this equation scalarly by $\vec{b}$, we get

$$([\vec{a},\vec{x}],\vec{b})+([\vec{x},\vec{b}],\vec{b})=([\vec{a},\vec{b}],\vec{b}) \quad \text{or} \quad (\vec{a},\vec{x},\vec{b})+(\vec{x},\vec{b},\vec{b})=(\vec{a},\vec{b},\vec{b}).$$

According to the properties of the mixed product $(\vec{x},\vec{b},\vec{b})=(\vec{a},\vec{b},\vec{b})=0$, that is $(\vec{a},\vec{x},\vec{b})=0$. This means that vectors $\vec{a}$, $\vec{x}$ and $\vec{b}$ are coplanar and linearly dependent

In this case, the vector $\vec{x}$ can be represented as a linear combination of vectors $\vec{a}$ and $\vec{b}$. Therefore, $\vec{x}=\alpha\vec{a}+\beta\vec{b}$.

Now we find at what values of α and β the vector $\vec{x}=\alpha\vec{a}+\beta\vec{b}$ will satisfy the original relation. Substituting, we get

$$[\vec{a},\vec{x}]+[\vec{x},\vec{b}]=[\vec{a},\alpha\vec{a}+\beta\vec{b}]+[\alpha\vec{a}+\beta\vec{b},\vec{b}]=$$

$$=\alpha[\vec{a},\vec{a}]+\beta[\vec{a},\vec{b}]+\alpha[\vec{a},\vec{b}]+\beta[\vec{b},\vec{b}]=(\alpha+\beta)[\vec{a},\vec{b}]=[\vec{a},\vec{b}],$$

that is, it is necessary that $\alpha+\beta=1$. Therefore $\vec{x}=\alpha\vec{a}+(1-\alpha)\vec{b}$, $\forall\alpha$.

Solution is found

Task 4.06

Find vector $\vec{x}$ *from a system of equations*

$$\begin{cases} [\vec{a},\vec{x}] = \vec{b}, \\ (\vec{c},\vec{x}) = \alpha, \end{cases}.$$

subject to $(\vec{c},\vec{a}) \neq 0$

Solution

We multiply both sides of the first equation vectorially from the left by $\vec{c}$. Then we use the property of double vector product. We get

$$[\vec{c},[\vec{a},\vec{x}]] = \vec{a}(\vec{c},\vec{x}) - \vec{x}(\vec{c},\vec{a}) = [\vec{c},\vec{b}]$$

$$\alpha\,\vec{a} - \vec{x}(\vec{c},\vec{a}) = [\vec{c},\vec{b}],$$

since due to the second equation of the system there will be $(\vec{c},\vec{x}) = \alpha$.

Where we finally get

$$\vec{x} = \frac{\alpha\,\vec{a} - [\vec{c},\vec{b}]}{(\vec{c},\vec{a})}.$$

Solution is found.

Vector Products in Coordinates

Expression of Scalar Product in Coordinates

Let a basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ and two vectors $\vec{a}$ and $\vec{b}$ with coordinate expansions in this basis:

$$\vec{a} = \xi_1\vec{g}_1 + \xi_2\vec{g}_2 + \xi_3\vec{g}_3\,, \qquad \vec{b} = \eta_1\vec{g}_1 + \eta_2\vec{g}_2 + \eta_3\vec{g}_3\,.$$

Then by properties 3) and 4) of the scalar product

$$\begin{aligned}(\vec{a},\vec{b}\,) &= (\,\xi_1\vec{g}_1 + \xi_2\vec{g}_2 + \xi_3\vec{g}_3,\ \eta_1\vec{g}_1 + \eta_2\vec{g}_2 + \eta_3\vec{g}_3\,) = \\ &= \xi_1\eta_1(\vec{g}_1,\vec{g}_1) + \xi_1\eta_2(\vec{g}_1,\vec{g}_2) + \xi_1\eta_3(\vec{g}_1,\vec{g}_3) + \\ &+ \xi_2\eta_1(\vec{g}_2,\vec{g}_1) + \xi_2\eta_2(\vec{g}_2,\vec{g}_2) + \xi_2\eta_3(\vec{g}_2,\vec{g}_3) + \\ &+ \xi_3\eta_1(\vec{g}_3,\vec{g}_1) + \xi_3\eta_2(\vec{g}_3,\vec{g}_2) + \xi_3\eta_3(\vec{g}_3,\vec{g}_3) = \\ &= \sum_{j=1}^{3}\Big(\xi_j\eta_1(\vec{g}_j,\vec{g}_1) + \xi_j\eta_2(\vec{g}_j,\vec{g}_2) + \xi_j\eta_3(\vec{g}_j,\vec{g}_3)\Big) = \\ &= \sum_{j=1}^{3}\sum_{i=1}^{3}\xi_j\eta_i(\vec{g}_j,\vec{g}_i)\,.\end{aligned}$$

In the case of an *orthonormal* basis, this formula is simplified, since for pairwise scalar products of basis vectors the equality $(\vec{e}_j, \vec{e}_i) = \delta_{ji} = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases}$ where the number δ_{ji} is called the *Kronecker delta.*

From where for the scalar product of vectors in an orthonormal basis we obtain the formula

$$(\vec{a}, \vec{b}) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 \,,$$

from which the following relations follow: $\left|\vec{a}\right| = \sqrt{(\vec{a}, \vec{a})} = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$, and for the case $\vec{a} \neq \vec{o}$ And $\vec{b} \neq \vec{o}$,

$$\cos\varphi = \frac{(\vec{a}, \vec{b})}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \cdot \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2}} \,.$$

Note that this equality in combination with the condition $|\cos\varphi| \leq 1$ leads to the correct $\forall \xi_i, \eta_j \quad i, j = 1 \ldots 3$ *Cauchy – Bunyakovsky inequality*:

$$|\,\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3\,| \leq \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \cdot \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2} \,.$$

Task 4.07 *Find the distance between two points in an Cartesian coordinate system with orthonormal basis if the coordinates of these points are known.*

Solution. Let the following be given: orthonormal coordinate system $\{\, O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and two points M_1 and M_2, whose radii=vectors have coordinate representations of the form

$$\left\| \vec{OM_1} \right\|_e = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| \qquad \text{and} \qquad \left\| \vec{OM_2} \right\|_e = \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\| .$$

Then, using the formula for the length of the vector

$$\vec{M_1M_2} = (\xi_1 - \eta_1)\vec{e}_1 + (\xi_2 - \eta_2)\vec{e}_2 + (\xi_3 - \eta_3)\vec{e}_3\,,$$

we obtain in the orthonormal coordinate system

Solution is found.

$$\left| \vec{M_1M_2} \right| = \sqrt{\,(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2\,}\;.$$

Expression of vector product in coordinates

Let *right* basis $\{\vec{g}_1,\ \vec{g}_2,\ \vec{g}_3\}$ be given (i.e. such that vectors $\{\vec{g}_1,\ \vec{g}_2,\ \vec{g}_3\}$ form *right-handed* triple) and let in this basis vectors $\vec{a}$ and $\vec{b}$ have coordinate decompositions

$$\vec{a} = \xi_1\vec{g}_1 + \xi_2\vec{g}_2 + \xi_3\vec{g}_3 \qquad \text{and} \qquad \vec{b} = \eta_1\vec{g}_1 + \eta_2\vec{g}_2 + \eta_3\vec{g}_3\,.$$

By properties 2) and 3) of the vector product we have

$$\begin{aligned}[\vec{a},\vec{b}\,] &= [\,\xi_1\vec{g}_1 + \xi_2\vec{g}_2 + \xi_3\vec{g}_3,\ \eta_1\vec{g}_1 + \eta_2\vec{g}_2 + \eta_3\vec{g}_3\,] = \\ &= \xi_1\eta_1[\vec{g}_1,\vec{g}_1] + \xi_1\eta_2[\vec{g}_1,\vec{g}_2] + \xi_1\eta_3[\vec{g}_1,\vec{g}_3] + \\ &+ \xi_2\eta_1[\vec{g}_2,\vec{g}_1] + \xi_2\eta_2[\vec{g}_2,\vec{g}_2] + \xi_2\eta_3[\vec{g}_2,\vec{g}_3] + \\ &+ \xi_3\eta_1[\vec{g}_3,\vec{g}_1] + \xi_3\eta_2[\vec{g}_3,\vec{g}_2] + \xi_3\eta_3[\vec{g}_3,\vec{g}_3] = \\ &= \sum_{j=1}^{3}\sum_{i=1}^{3}\xi_j\eta_i[\vec{g}_j,\vec{g}_i]\,.\end{aligned}$$

If we introduce the notation

$$\vec{f}_1 = [\vec{g}_2, \vec{g}_3] , \quad \vec{f}_2 = [\vec{g}_3, \vec{g}_1] , \quad \vec{f}_3 = [\vec{g}_1, \vec{g}_2] ,$$

then we get an easy-to-remember formula

$$[\vec{a}, \vec{b}] = (\xi_2\eta_3 - \xi_3\eta_2)\, \vec{f}_1 - (\xi_1\eta_3 - \xi_3\eta_1)\, \vec{f}_2 + (\xi_1\eta_2 - \xi_2\eta_1)\, \vec{f}_3 =$$

$$= \vec{f}_1 \det \left\| \begin{matrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{matrix} \right\| - \vec{f}_2 \det \left\| \begin{matrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{matrix} \right\| + \vec{f}_3 \det \left\| \begin{matrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{matrix} \right\| =$$

$$= \det \left\| \begin{matrix} \vec{f}_1 & \vec{f}_2 & \vec{f}_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{matrix} \right\| .$$

The case of an orthonormal basis

Let the original basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ *orthonormal*, forming a *right* triple of vectors, then by the definition of the vector product

$$\vec{f}_1 = \vec{e}_1\,, \quad \vec{f}_2 = \vec{e}_2\,, \quad \vec{f}_3 = \vec{e}_3\,.$$

Then the formula for the vector product of vectors in the *right orthonormal* basis will be noticeably simplified:

$$[\vec{a}, \vec{b}] = \det \left\| \begin{matrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{matrix} \right\| .$$

The case of an orthonormal basis

The above formulas have useful consequences.

Corollary **For vectors $\vec{a}$ and $\vec{b}$ to be collinear, it is necessary and sufficient that in any basis**

$$\det\left\|\begin{matrix}\xi_2 & \xi_3\\ \eta_2 & \eta_3\end{matrix}\right\| = \det\left\|\begin{matrix}\xi_1 & \xi_3\\ \eta_1 & \eta_3\end{matrix}\right\| = \det\left\|\begin{matrix}\xi_1 & \xi_2\\ \eta_1 & \eta_2\end{matrix}\right\| = 0\,,$$

or (in the case of $\vec{b} \neq \vec{o}$) $\dfrac{\xi_1}{\eta_1} = \dfrac{\xi_2}{\eta_2} = \dfrac{\xi_3}{\eta_3}\,.$

Corollary **In an orthonormal basis, the area of a parallelogram constructed on the vectors $\vec{a}$ and $\vec{b}$, is**

$$S = \sqrt{\det^2\left\|\begin{matrix}\xi_2 & \xi_3\\ \eta_2 & \eta_3\end{matrix}\right\| + \det^2\left\|\begin{matrix}\xi_1 & \xi_3\\ \eta_1 & \eta_3\end{matrix}\right\| + \det^2\left\|\begin{matrix}\xi_1 & \xi_2\\ \eta_1 & \eta_2\end{matrix}\right\|}\,,$$

and for the case on the plane $S = \left|\det\left\|\begin{matrix}\xi_2 & \xi_3\\ \eta_2 & \eta_3\end{matrix}\right\|\right|\,.$

Expression of a mixed product in coordinates

Let a *right* basis be given $\{\vec{g}_1,\, \vec{g}_2,\, \vec{g}_3\}$ and let in this basis the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ have coordinate decompositions

$$\vec{a} = \xi_1\vec{g}_1 + \xi_2\vec{g}_2 + \xi_3\vec{g}_3\,, \qquad \vec{b} = \eta_1\vec{g}_1 + \eta_2\vec{g}_2 + \eta_3\vec{g}_3$$

and accordingly $\quad \vec{c} = \kappa_1\vec{g}_1 + \kappa_2\vec{g}_2 + \kappa_3\vec{g}_3\,.$

It was shown earlier that the vector product in coordinates can be represented as

$$[\vec{a}, \vec{b}] = \vec{f}_1 \det \left\| \begin{array}{cc} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{array} \right\| - \vec{f}_2 \det \left\| \begin{array}{cc} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{array} \right\| + \vec{f}_3 \det \left\| \begin{array}{cc} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{array} \right\| ,$$

where

$$\vec{f}_1 = [\vec{g}_2, \vec{g}_3]\,, \quad \vec{f}_2 = [\vec{g}_3, \vec{g}_1]\,, \quad \vec{f}_3 = [\vec{g}_1, \vec{g}_2]\,.$$

Using the definition of a mixed product, it is easy to verify that the last equalities imply the relations

$$(\vec{g}_k, \vec{f}_j) = \begin{cases} (\vec{g}_1,\, \vec{g}_2,\, \vec{g}_3) & \text{for} \quad k = j, \\ 0 & \text{for} \quad k \neq j \end{cases} \qquad \forall k, j = 1, 2, 3\,.$$

Then we get for the mixed product will be

$$(\vec{a}, \vec{b}, \vec{c}) = ([\vec{a}, \vec{b}], \vec{c}) = \left(\kappa_1 \det \begin{Vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{Vmatrix} - \right.$$

$$\left. -\kappa_2 \det \begin{Vmatrix} \xi_1 & \xi_3 \\ \eta_1 & \eta_3 \end{Vmatrix} + \kappa_3 \det \begin{Vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{Vmatrix} \right) (\vec{g}_1,\, \vec{g}_2,\, \vec{g}_3) =$$

$$= \det \begin{Vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{Vmatrix} (\vec{g}_1,\, \vec{g}_2,\, \vec{g}_3)\,.$$

Indeed, the expression in large parentheses is the expansion of the determinant of the 3rd order by its last line.

Notes: 1) In the case of a right orthonormal basis we have $(\vec{e}_1,\, \vec{e}_2,\, \vec{e}_3) = 1$. Therefore, in such a basis

$$(\vec{a}, \vec{b}, \vec{c}) = \det \left\| \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{array} \right\| .$$

2) For a triple of vectors $\left\{ \vec{f}_1,\, \vec{f}_2,\, \vec{f}_3 \right\}$ is true

Theorem **Triple of vectors $\left\{ \vec{f}_1,\, \vec{f}_2,\, \vec{f}_3 \right\}$ linearly independent.**

Corollary **The triple of vectors $\left\{ \vec{f}_1,\, \vec{f}_2,\, \vec{f}_3 \right\}$ forms a basis (called *reciprocal* to the basis $\{\vec{g}_1,\, \vec{g}_2,\, \vec{g}_3\}$) .**