

## **Basis. Vector coordinates in the basis**

Definition	<p>A <i>basis</i> on a line is any <i>nonzero</i> vector belonging to this line.</p> <p>A <i>basis</i> on a plane is any <i>ordered</i> pair of <i>linearly independent</i> vectors belonging to this plane.</p> <p>A <i>basis</i> in space is any <i>ordered</i> triple of <i>linearly independent</i> vectors.</p>
Definition	<p>A basis is called <i>orthogonal</i> if the vectors forming it are pairwise orthogonal (mutually perpendicular).</p>
Definition	<p>An orthogonal basis is called <i>orthonormal</i> if the vectors forming it have unit length.</p>

Theorem **Let a basis  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ , be given, then any vector  $\vec{x}$  in space can be represented, and uniquely, in the form**

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3,$$

**where  $\xi_1, \xi_2, \xi_3$  are some numbers.**

Definition The numbers  $\xi_1, \xi_2, \xi_3$  are the coefficients in the vector expansion  $\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$ . They are called the *coordinates* (or *components*) of the vector  $\vec{x}$  in the basis  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ . These numbers are usually written in the form of a column  $\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \vec{g} \end{pmatrix}$ , which is called the *coordinate column* or coordinate representation of the vector.

## Vector operations in coordinate representation

The rules for operations with vectors in coordinate form coincide with the rules for the corresponding operations with matrices.

The following holds

**Theorem** In coordinate representation, operations with vectors are performed as follows:

1°. *Equality of vectors* Two vectors

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$$

and

$$\vec{y} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3$$

are equal if and only if their coordinate representations are equal:

$$\left\| \vec{x} \right\|_g = \left\| \vec{y} \right\|_g \quad \text{or} \quad \begin{cases} \xi_1 = \eta_1 \\ \xi_2 = \eta_2 \\ \xi_3 = \eta_3 \end{cases} .$$

2°. *Vector addition*

**The coordinate representation of the sum of two vectors**

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$$

**and** 
$$\vec{y} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3$$

**is equal to the sum of the coordinate representations of the terms**

$$\left\| \vec{x} + \vec{y} \right\|_g = \left\| \vec{x} \right\|_g + \left\| \vec{y} \right\|_g .$$

3°. *Vector multiplication (by a number)*

**The coordinate representation of the product of a number  $\lambda$  and a vector**

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$$

**is equal to the product of a number  $\lambda$  and the coordinate representation of a vector  $\vec{x}$  :**

$$\left\| \lambda \vec{x} \right\|_g = \lambda \left\| \vec{x} \right\|_g .$$

Proof.

Let us consider the rule of vector addition in coordinate form.

$$\begin{aligned} \left\| \begin{matrix} \vec{x} \\ \vec{y} \end{matrix} \right\|_g &= \left\| (\xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3) + (\eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3) \right\|_g = \\ &= \left\| (\xi_1 + \eta_1) \vec{g}_1 + (\xi_2 + \eta_2) \vec{g}_2 + (\xi_3 + \eta_3) \vec{g}_3 \right\|_g = \\ &= \left\| \begin{matrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \xi_3 + \eta_3 \end{matrix} \right\| = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| + \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\| = \left\| \vec{x} \right\|_g + \left\| \vec{y} \right\|_g. \end{aligned}$$

The theorem is proved.

**Corollary** The coordinate representation of a linear combination  $\lambda \vec{x} + \mu \vec{y}$  is the same linear combination of the coordinate representations of vectors  $\vec{x}$  and  $\vec{y}$ :

$$\left\| \begin{matrix} \lambda \xi_1 + \mu \eta_1 \\ \lambda \xi_2 + \mu \eta_2 \\ \lambda \xi_3 + \mu \eta_3 \end{matrix} \right\| = \lambda \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| + \mu \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\|.$$

Let us now consider the question: how the conditions of linear dependence and independence of vectors are written in the coordinate representation?

Theorem **In order for two vectors  $\vec{x}$  and  $\vec{y}$  on the plane to be linearly dependent, it is necessary and sufficient that their coordinate representations  $\left\| \vec{x} \right\|_g = \left\| \begin{matrix} \xi_1 \\ \xi_2 \end{matrix} \right\|$  and**

**$\left\| \vec{y} \right\|_g = \left\| \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right\|$  satisfy the condition**

$$\det \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} = 0.$$

Theorem **In order for three vectors in space  $\{\vec{x}, \vec{y}, \vec{z}\}$  with coordinate representations**

$$\left\| \vec{x} \right\|_g = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\|, \quad \left\| \vec{y} \right\|_g = \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\| \quad \text{and} \quad \left\| \vec{z} \right\|_g = \left\| \begin{matrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{matrix} \right\|$$

**to be linearly dependent, it is necessary and sufficient that their coordinates satisfy the condition**

$$\det \begin{vmatrix} \xi_1 & \eta_1 & \kappa_1 \\ \xi_2 & \eta_2 & \kappa_2 \\ \xi_3 & \eta_3 & \kappa_3 \end{vmatrix} = 0.$$

Task

1) *Will the columns be linearly dependent*

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} ?$$

(Ans. *No*)

2) *What values of the parameter  $a$  will the columns be linearly dependent*

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ a \end{pmatrix} ?$$

(Ans. *At  $a = 1$* )

## Cartesian coordinate system

Definition The set of the basis  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$  and the point  $O$  in which the origins of all basis vectors are placed is called the general *Cartesian coordinate system* and is denoted by  $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ .

Definition The coordinate system  $\{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$  generated by the orthonormal basis is called the *normal rectangular* (or *orthonormal*) coordinate system.

If the coordinate system  $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$  is given, then an arbitrary point  $M$  in space can be put into one-to-one correspondence with the vector  $\vec{r}$ , the origin of which is at the point  $O$  and the end is at the point  $M$ .

Definition The vector  $\vec{r} = \vec{OM}$  is called the *position vector* of the point  $M$  in the coordinate system  $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ .

Definition The coordinates of the position vector of the point  $M$  are called the *coordinates of the point  $M$*  in the coordinate system  $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ .

**Changing coordinates when replacing the basis and the origin**

Let two Cartesian coordinate systems be given: “old”  $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$  and “new”  $\{O', \vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$ . Let us express the vectors of the “new” basis, as well as the vector  $\vec{OO'}$  through the vectors of the “old” basis. Due to the properties of the basis, this can always be done in a unique way:

$$\begin{aligned} \vec{g}'_1 &= \sigma_{11} \vec{g}_1 + \sigma_{21} \vec{g}_2 + \sigma_{31} \vec{g}_3, \\ \vec{g}'_2 &= \sigma_{12} \vec{g}_1 + \sigma_{22} \vec{g}_2 + \sigma_{32} \vec{g}_3, \\ \vec{g}'_3 &= \sigma_{13} \vec{g}_1 + \sigma_{23} \vec{g}_2 + \sigma_{33} \vec{g}_3, \\ \vec{OO'} &= \beta_1 \vec{g}_1 + \beta_2 \vec{g}_2 + \beta_3 \vec{g}_3. \end{aligned} \tag{1}$$

Then the following holds:

**Theorem** The coordinates of an arbitrary point in the “old” coordinate system are related to its coordinates in the “new” by the relations

$$\begin{aligned} \xi_1 &= \sigma_{11} \xi'_1 + \sigma_{12} \xi'_2 + \sigma_{13} \xi'_3 + \beta_1, \\ \xi_2 &= \sigma_{21} \xi'_1 + \sigma_{22} \xi'_2 + \sigma_{23} \xi'_3 + \beta_2, \\ \xi_3 &= \sigma_{31} \xi'_1 + \sigma_{32} \xi'_2 + \sigma_{33} \xi'_3 + \beta_3. \end{aligned} \tag{2}$$

**Definition** Formulas (2) are called *formulas for the transition* from a coordinate system  $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$  to a coordinate system  $\{O', \vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$ .

Definition The matrix  $\|S\| = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$  is called the basis  $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$ -to-basis  $\{\vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$  transition matrix.

Theorem **For a transition matrix**

$$\det \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \neq 0.$$

Task *Write the formulas for the direct and inverse transition for two Cartesian coordinate systems shown in Fig. 1.*

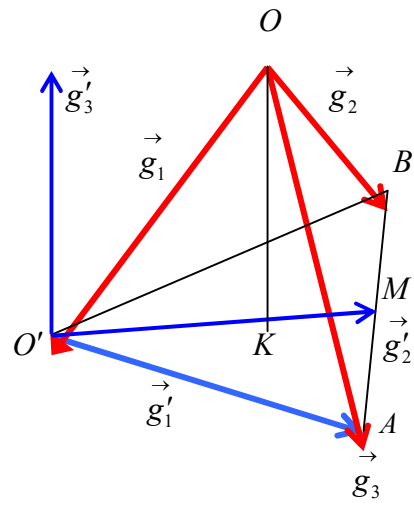


Fig. 1.

*Here  $OO'AB$  is a regular triangular pyramid, point  $M$  is the midpoint of  $AB$  and  $OK$  is the height of the pyramid.*

Solution

Let us find the formulas for the transition from the coordinate system  $\{\vec{O}, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$  to  $\{\vec{O}', \vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$ . We have from Fig. 1.  $\vec{OO}' = \vec{g}_1$ . And for the "new" basis vectors

$$\begin{aligned}\vec{g}'_1 &= -\vec{g}_1 + \vec{g}_3 \\ \vec{g}'_2 &= -\vec{g}_1 + \frac{\vec{g}_2 + \vec{g}_3}{2} \\ \vec{g}'_3 &= -\vec{g}_1 + \vec{O}'K = -\vec{g}_1 - \frac{2}{3}\vec{g}'_2 = -\frac{1}{3}\vec{g}_1 - \frac{1}{3}\vec{g}_2 - \frac{1}{3}\vec{g}_3.\end{aligned}$$

Having written down in columns the found coordinate decompositions of the "new" basis vectors by the "old" ones, we obtain the transition matrix

$$\|S\| = \begin{vmatrix} -1 & -1 & -\frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{3} \end{vmatrix},$$

whose determinant is equal to  $\frac{1}{2}$ . Now we write down the formulas for the *direct transition*

$$\begin{cases} \xi_1 = -\xi'_1 - \xi'_2 - \frac{1}{3}\xi'_3 + 1 \\ \xi_2 = \frac{1}{2}\xi'_2 - \frac{1}{3}\xi'_3 \\ \xi_3 = \xi'_1 + \frac{1}{2}\xi'_2 - \frac{1}{3}\xi'_3 \end{cases}.$$

Now let's find the formulas for the inverse transition. To do this, we first express the vectors of the "old" basis through the vectors of the "new".

$$\vec{g}_1 = -\frac{2}{3}\vec{g}'_2 - \vec{g}'_3$$

$$\vec{g}_2 = \vec{OM} + \vec{MB} = (\vec{KM} - \vec{g}'_3) + (\vec{g}'_2 - \vec{g}'_1) = -\vec{g}'_1 + \frac{4}{3}\vec{g}'_2 - \vec{g}'_3$$

$$\vec{g}_3 = \vec{g}_1 + \vec{g}'_1 = \vec{g}'_1 - \frac{2}{3}\vec{g}'_2 - \vec{g}'_3.$$

Then the matrix of the inverse transition will have the form

$$\|T\| = \begin{vmatrix} 0 & -1 & 1 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ 1 & -1 & -1 \end{vmatrix},$$

and  $\det\|T\| = 2$ . Finally, the formulas for the inverse transition will be

$$\begin{cases} \xi'_1 = & -\xi_2 & +\xi_3 \\ \xi'_2 = & -\frac{2}{3}\xi_1 + \frac{4}{3}\xi_2 - \frac{2}{3}\xi_3 + \frac{2}{3} \\ \xi'_3 = & -\xi_1 & -\xi_2 & -\xi_3 + 1 \end{cases}$$

$$\text{since } \vec{O'O} = -\vec{g}_1 = \frac{2}{3}\vec{g}'_2 + \vec{g}'_3.$$

Solution is found

## Transition formulas between orthonormal coordinate systems on a plane

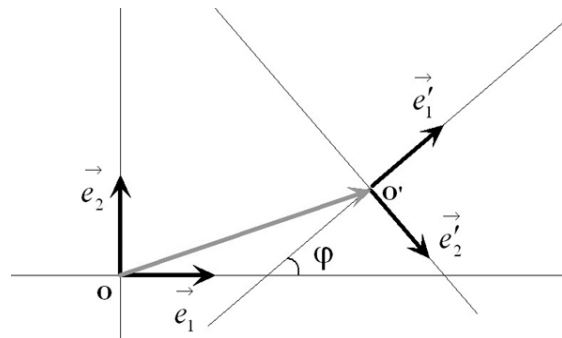
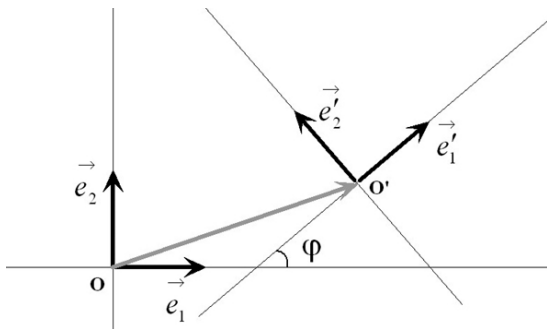
Let us consider two orthonormal coordinate systems  $\{O, \vec{e}_1, \vec{e}_2\}$  and  $\{O', \vec{e}'_1, \vec{e}'_2\}$ . We obtain the transition formulas for the case shown in the figure. From the geometrically obvious relations

$$\vec{e}'_1 = \vec{e}_1 \cos \varphi + \vec{e}_2 \sin \varphi \quad \text{and} \quad \vec{e}'_2 = -\vec{e}_1 \sin \varphi + \vec{e}_2 \cos \varphi$$

we obtain the transition matrix: 
$$\|S\| = \begin{vmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{vmatrix},$$

and if  $\vec{OO}' = \begin{vmatrix} \beta_1 \\ \beta_2 \end{vmatrix}$ , then the “old” coordinates will be related to the “new” ones as

$$\begin{cases} \xi_1 = \xi'_1 \cos \varphi - \xi'_2 \sin \varphi + \beta_1, \\ \xi_2 = \xi'_1 \sin \varphi + \xi'_2 \cos \varphi + \beta_2. \end{cases}$$



In the first case, both coordinate systems can be combined by successively performing a parallel transfer of the “old” system by a vector  $\vec{OO'}$  and a rotation by an angle  $\varphi$  around a point  $O'$ .

Sometimes, after combining the vectors  $\vec{e}_1$  and  $\vec{e}'_1$ , it will also be necessary to reflect the vector  $\vec{e}_2$  symmetrically with respect to a straight line passing through the combined vectors. The transition formulas in this case will have the form

$$\begin{cases} \xi_1 = \xi'_1 \cos \varphi + \xi'_2 \sin \varphi + \beta_1, \\ \xi_2 = \xi'_1 \sin \varphi - \xi'_2 \cos \varphi + \beta_2. \end{cases}$$