Basis. Vector coordinates in the basis

Definition	A *basis* on a line is any *nonzero* vector belonging to this line. A *basis* on a plane is any *ordered* pair of *linearly independent* vectors belonging to this plane. A *basis* in space is any *ordered* triple of *linearly independent* vectors.
Definition	A basis is called *orthogonal* if the vectors forming it are pairwise orthogonal (mutually perpendicular).
Definition	An orthogonal basis is called *orthonormal* if the vectors forming it have unit length.

Theorem **Let a basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$, be given, then any vector $\vec{x}$ in space can be represented, and uniquely, in the form**

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3 ,$$

where ξ_1, ξ_2, ξ_3 are some numbers.

Definition The numbers ξ_1, ξ_2, ξ_3 are the coefficients in the vector expansion $\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$. They are called the *coordinates* (or *components*) of the vector $\vec{x}$ in the basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$. These numbers are usually written in the form of a column $\begin{Vmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Vmatrix} = \left\| \vec{x} \right\|_g$, which is called the *coordinate column* or coordinate representation of the vector.

Vector operations in coordinate representation

The rules for operations with vectors in coordinate form coincide with the rules for the corresponding operations with matrices.

The following holds

Theorem **In coordinate representation, operations with vectors are performed as follows:**

1°. ***Equality of vectors*** **Two vectors**

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$$

and $$\vec{y} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3$$

are equal if and only if their coordinate representations are equal:

$$\left\| \vec{x} \right\|_g = \left\| \vec{y} \right\|_g \quad \textbf{or} \quad \begin{cases} \xi_1 = \eta_1 \\ \xi_2 = \eta_2 \\ \xi_3 = \eta_3 \end{cases}.$$

2°. ***Vector addition***

The coordinate representation of the sum of two vectors

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$$

and $$\vec{y} = \eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3$$

is equal to the sum of the coordinate representations of the terms

$$\left\| \vec{x} + \vec{y} \right\|_g = \left\| \vec{x} \right\|_g + \left\| \vec{y} \right\|_g .$$

3°. ***Vector multiplication (by a numbe***)

The coordinate representation of the product of a number λ **and a vector**

$$\vec{x} = \xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3$$

is equal to the product of a number λ **and the coordinate representation of a vector** $\vec{x}$ **:**

$$\left\| \lambda \vec{x} \right\|_g = \lambda \left\| \vec{x} \right\|_g .$$

Proof.

Let us consider the rule of vector addition in coordinate form.

$$\left\| \vec{x}+\vec{y} \right\|_g = \left\| (\xi_1 \vec{g}_1 + \xi_2 \vec{g}_2 + \xi_3 \vec{g}_3) + (\eta_1 \vec{g}_1 + \eta_2 \vec{g}_2 + \eta_3 \vec{g}_3) \right\|_g =$$

$$= \left\| (\xi_1+\eta_1)\vec{g}_1 + (\xi_2+\eta_2)\vec{g}_2 + (\xi_3+\eta_3)\vec{g}_3 \right\|_g =$$

$$= \left\| \begin{matrix} \xi_1+\eta_1 \\ \xi_2+\eta_2 \\ \xi_3+\eta_3 \end{matrix} \right\| = \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| + \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\| = \left\| \vec{x} \right\|_g + \left\| \vec{y} \right\|_g .$$

The theorem is proved.

Corollary **The coordinate representation of a linear combination $\lambda \vec{x} + \mu \vec{y}$ is the same linear combination of the coordinate representations of vectors $\vec{x}$ and $\vec{y}$:**

$$\left\| \begin{matrix} \lambda\xi_1+\mu\eta_1 \\ \lambda\xi_2+\mu\eta_2 \\ \lambda\xi_3+\mu\eta_3 \end{matrix} \right\| = \lambda \left\| \begin{matrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{matrix} \right\| + \mu \left\| \begin{matrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{matrix} \right\| .$$

Let us now consider the question: how the conditions of linear dependence and independence of vectors are written in the coordinate representation?

Theorem **In order for two vectors $\vec{x}$ and $\vec{y}$ on the plane to be linearly dependent, it is necessary and sufficient that their coordinate representations $\left\|\vec{x}\right\|_g = \left\|\begin{matrix}\xi_1\\ \xi_2\end{matrix}\right\|$ and $\left\|\vec{y}\right\|_g = \left\|\begin{matrix}\eta_1\\ \eta_2\end{matrix}\right\|$ satisfy the condition**

$$\det\left\|\begin{matrix}\xi_1 & \eta_1\\ \xi_2 & \eta_2\end{matrix}\right\| = 0.$$

Theorem **In order for three vectors in space $\{\vec{x}, \vec{y}, \vec{z}\}$ with coordinate representations**

$$\left\|\vec{x}\right\|_g = \left\|\begin{matrix}\xi_1\\ \xi_2\\ \xi_3\end{matrix}\right\|, \quad \left\|\vec{y}\right\|_g = \left\|\begin{matrix}\eta_1\\ \eta_2\\ \eta_3\end{matrix}\right\| \quad \text{and} \quad \left\|\vec{z}\right\|_g = \left\|\begin{matrix}\kappa_1\\ \kappa_2\\ \kappa_3\end{matrix}\right\|$$

to be linearly dependent, it is necessary and sufficient that their coordinates satisfy the condition

$$\det\left\|\begin{matrix}\xi_1 & \eta_1 & \kappa_1\\ \xi_2 & \eta_2 & \kappa_2\\ \xi_3 & \eta_3 & \kappa_3\end{matrix}\right\| = 0.$$

Task

1) *Will the columns be linearly dependent*

$$\left\|\begin{matrix}3\\1\\2\end{matrix}\right\|,\quad \left\|\begin{matrix}1\\-1\\1\end{matrix}\right\|,\quad \left\|\begin{matrix}2\\2\\0\end{matrix}\right\|?$$

(Ans. *No*)

2) *What values of the parameter* a *will the columns be linearly dependent*

$$\left\|\begin{matrix}3\\1\\2\end{matrix}\right\|,\quad \left\|\begin{matrix}1\\-1\\1\end{matrix}\right\|,\quad \left\|\begin{matrix}2\\2\\a\end{matrix}\right\|?$$

(Ans. *At* $a=1$)

Cartesian coordinate system

Definition.	The set of the basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$. and the point O in which the origins of all basis vectors are placed is called the general *Cartesian coordinate system* and is denoted by $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$.
Definition	The coordinate system $\{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ generated by the orthonormal basis is called the *normal rectangular* (or *orthonormal*) coordinate system.

If the coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ is given, then an arbitrary point M in space can be put into one-to-one correspondence with the vector $\vec{r}$, the origin of which is at the point O and the end is at the point M.

Definition	The vector $\vec{r} = \vec{OM}$ is called the *position vector* of the point M in the coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$.
Definition	The coordinates of the position vector of the point M are called the *coordinates of the point* M in the coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$.

Changing coordinates when replacing the basis and the origin

Let two Cartesian coordinate systems be given: "old" $\{O,\vec{g}_1,\vec{g}_2,\vec{g}_3\}$ and "new" $\{O',\vec{g}'_1,\vec{g}'_2,\vec{g}'_3\}$. Let us express the vectors of the "new" basis, as well as the vector $\vec{OO'}$ through the vectors of the "old" basis. Due to the properties of the basis, this can always be done in a unique way:

$$\begin{aligned}
\vec{g}'_1 &= \sigma_{11}\vec{g}_1+\sigma_{21}\vec{g}_2+\sigma_{31}\vec{g}_3,\\
\vec{g}'_2 &= \sigma_{12}\vec{g}_1+\sigma_{22}\vec{g}_2+\sigma_{32}\vec{g}_3,\\
\vec{g}'_3 &= \sigma_{13}\vec{g}_1+\sigma_{23}\vec{g}_2+\sigma_{33}\vec{g}_3,\\
\vec{OO'} &= \beta_1\vec{g}_1+\beta_2\vec{g}_2+\beta_3\vec{g}_3.
\end{aligned} \tag{1}$$

Then the following holds:

Theorem **The coordinates of an arbitrary point in the "old" coordinate system are related to its coordinates in the "new" by the relations**

$$\begin{aligned}
\xi_1 &= \sigma_{11}\xi'_1+\sigma_{12}\xi'_2+\sigma_{13}\xi'_3+\beta_1,\\
\xi_2 &= \sigma_{21}\xi'_1+\sigma_{22}\xi'_2+\sigma_{23}\xi'_3+\beta_2,\\
\xi_3 &= \sigma_{31}\xi'_1+\sigma_{32}\xi'_2+\sigma_{33}\xi'_3+\beta_3.
\end{aligned} \tag{2}$$

Definition Formulas (2) are called *formulas for the transition* from a coordinate system $\{O,\vec{g}_1,\vec{g}_2,\vec{g}_3\}$ to a coordinate system $\{O',\vec{g}'_1,\vec{g}'_2,\vec{g}'_3\}$.

Definition The matrix $\| S \| = \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{Vmatrix}$ is called the basis $\{\vec{g}_1, \vec{g}_2, \vec{g}_3\}$-to-basis $\{\vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$ *transition matrix.*

Theorem **For a transition matrix**

$$\det \begin{Vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{Vmatrix} \neq 0.$$

Task *Write the formulas for the direct and inverse transition for two Cartesian coordinate systems shown in Fig. 1.*

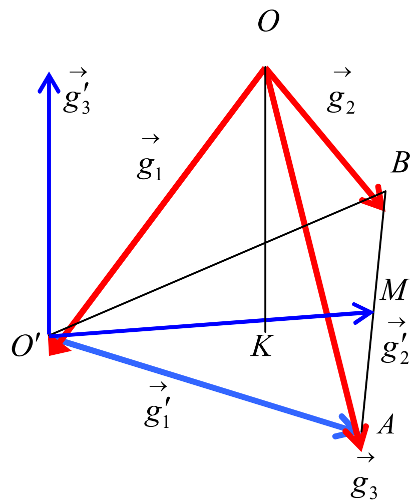


Fig. 1.

Here $OO'AB$ is a regular triangular pyramid, point M is the midpoint of AB and OK is the height of the pyramid.

Solution

Let us find the formulas for the transition from the coordinate system $\{O, \vec{g}_1, \vec{g}_2, \vec{g}_3\}$ to $\{O', \vec{g}'_1, \vec{g}'_2, \vec{g}'_3\}$. We have from Fig. 1. $\vec{OO'} = \vec{g}_1$. And for the "new" basis vectors

$$\begin{aligned} \vec{g}'_1 &= -\vec{g}_1 + \vec{g}_3 \\ \vec{g}'_2 &= -\vec{g}_1 + \frac{\vec{g}_2 + \vec{g}_3}{2} \\ \vec{g}'_3 &= -\vec{g}_1 + \vec{O'K} = -\vec{g}_1 - \frac{2}{3}\vec{g}'_2 = -\frac{1}{3}\vec{g}_1 - \frac{1}{3}\vec{g}_2 - \frac{1}{3}\vec{g}_3 . \end{aligned}$$

Having written down in columns the found coordinate decompositions of the "new" basis vectors by the "old" ones, we obtain the transition matrix

$$\| S \| = \left\| \begin{matrix} -1 & -1 & -\frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{3} \end{matrix} \right\| ,$$

whose determinant is equal to $\frac{1}{2}$. Now we write down the formulas for the *direct transition*

$$\begin{cases} \xi_1 = -\xi'_1 - \xi'_2 - \frac{1}{3}\xi'_3 + 1 \\ \xi_2 = \frac{1}{2}\xi'_2 - \frac{1}{3}\xi'_3 \\ \xi_3 = \xi'_1 + \frac{1}{2}\xi'_2 - \frac{1}{3}\xi'_3 \end{cases} .$$

Now let's find the formulas for the inverse transition. To do this, we first express the vectors of the "old" basis through the vectors of the "new".

$$\vec{g}_1 = -\frac{2}{3}\vec{g}'_2 - \vec{g}'_3$$
$$\vec{g}_2 = \vec{OM} + \vec{MB} = (\vec{KM} - \vec{g}'_3) + (\vec{g}'_2 - \vec{g}'_1) = -\vec{g}'_1 + \frac{4}{3}\vec{g}'_2 - \vec{g}'_3$$
$$\vec{g}_3 = \vec{g}_1 + \vec{g}'_1 = \vec{g}'_1 - \frac{2}{3}\vec{g}'_2 - \vec{g}'_3 .$$

Then the matrix of the inverse transition will have the form

$$\| T \| = \begin{Vmatrix} 0 & -1 & 1 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ -1 & -1 & -1 \end{Vmatrix},$$

and $\det\| T \| = 2$. Finally, the formulas for the inverse transition will be

$$\begin{cases} \xi'_1 = & -\xi_2 + \xi_3 \\ \xi'_2 = -\frac{2}{3}\xi_1 + \frac{4}{3}\xi_2 - \frac{2}{3}\xi_3 + \frac{2}{3} \\ \xi'_3 = -\xi_1 - \xi_2 - \xi_3 + 1 \end{cases}$$

since $\vec{O'O} = -\vec{g}_1 = \frac{2}{3}\vec{g}'_2 + \vec{g}'_3$.

Solution is found

Transition formulas between orthonormal coordinate systems on a plane

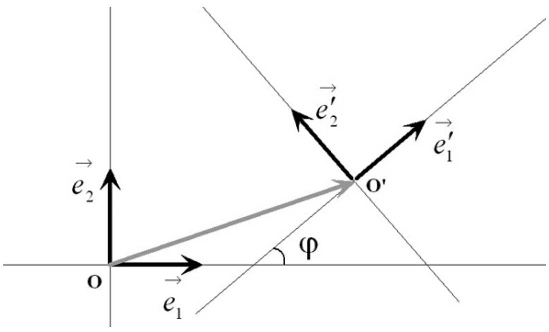
Let us consider two orthonormal coordinate systems $\{O, \vec{e}_1, \vec{e}_2\}$ and $\{O', \vec{e}'_1, \vec{e}'_2\}$. We obtain the transition formulas for the case shown in the figure. From the geometrically obvious relations

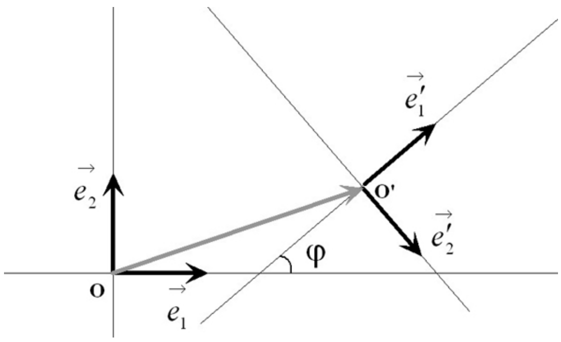
$$\vec{e}'_1 = \vec{e}_1 \cos\varphi + \vec{e}_2 \sin\varphi \quad \text{and} \quad \vec{e}'_2 = -\vec{e}_1 \sin\varphi + \vec{e}_2 \cos\varphi$$

we obtain the transition matrix: $\quad \| S \| = \begin{Vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{Vmatrix},$

and if $\vec{OO'} = \begin{Vmatrix} \beta_1 \\ \beta_2 \end{Vmatrix}$, then the “old” coordinates will be related to the “new” ones as

$$\begin{cases} \xi_1 = \xi'_1 \cos\varphi - \xi'_2 \sin\varphi + \beta_1, \\ \xi_2 = \xi'_1 \sin\varphi + \xi'_2 \cos\varphi + \beta_2. \end{cases}$$





In the first case, both coordinate systems can be combined by successively performing a parallel transfer of the "old" system by a vector $\overrightarrow{OO'}$ and a rotation by an angle φ around a point O'.

Sometimes, after combining the vectors $\vec{e}_1$ and $\vec{e}'_1$, it will also be necessary to reflect the vector $\vec{e}_2$ symmetrically with respect to a straight line passing through the combined vectors. The transition formulas in this case will have the form

$$\begin{cases} \xi_1 = \xi'_1 \cos\varphi + \xi'_2 \sin\varphi + \beta_1, \\ \xi_2 = \xi'_1 \sin\varphi - \xi'_2 \cos\varphi + \beta_2. \end{cases}$$